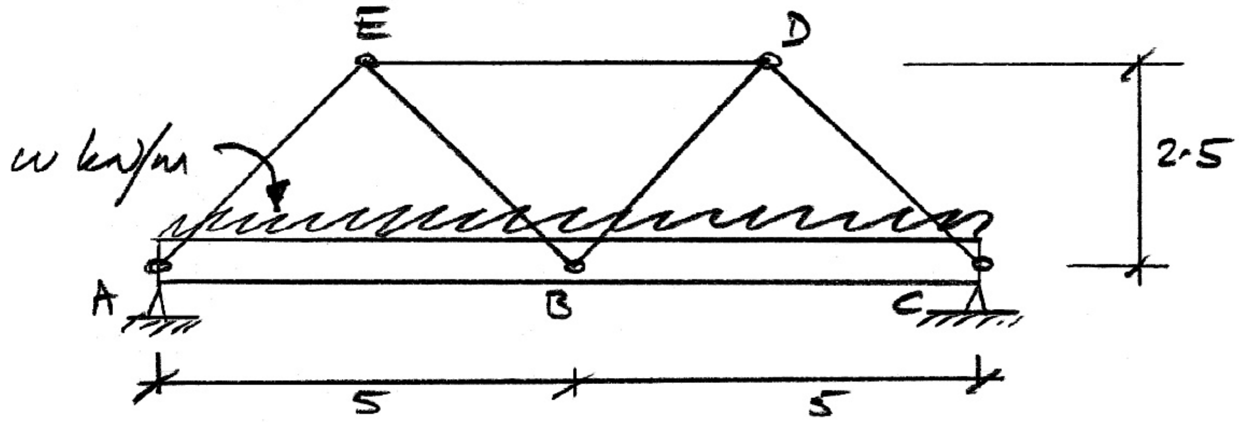
Chapter 2 - Virtual Work: Compound Structures

Rev. 1

2.1 Introduction

2.1.1 Purpose

Previously we only used virtual work to analyse structures whose members primarily behaved in flexure or in axial forces. Many real structures are comprised of a mixture of such members. Cable-stay and suspension bridges area good examples: the deck-level carries load primarily through bending whilst the cable and pylon elements carry load through axial forces mainly. A simple example is a trussed beam:



Other structures carry load through a mixture of bending, axial force, torsion, etc. Our knowledge of virtual work to-date is sufficient to analyse such structures.

2.2 Virtual Work Development

2.2.1 The Principle of Virtual Work

This states that:

> *A body is in equilibrium if, and only if, the virtual work of all forces acting on the body is zero.*

In this context, the word 'virtual' means 'having the effect of, but not the actual form of, what is specified'.

There are two ways to define virtual work, as follows.

1. Virtual Displacement:
   Virtual work is the work done by the actual forces acting on the body moving through a virtual displacement.

2. Virtual Force:
   Virtual work is the work done by a virtual force acting on the body moving through the actual displacements.

Virtual Displacements

A virtual displacement is a displacement that is only imagined to occur:

- virtual displacements must be small enough such that the force directions are maintained.

- virtual displacements within a body must be geometrically compatible with the original structure. That is, geometrical constraints (i.e. supports) and member continuity must be maintained.

Virtual Forces

A virtual force is a force imagined to be applied and is then moved through the actual deformations of the body, thus causing virtual work.

Virtual forces must form an equilibrium set of their own.

Internal and External Virtual Work

When a structures deforms, work is done both by the applied loads moving through a displacement, as well as by the increase in strain energy in the structure. Thus when virtual displacements or forces are causing virtual work, we have:

$$\begin{aligned} \delta W &= 0 \\ \delta W_I - \delta W_E &= 0 \\ \delta W_E &= \delta W_I \end{aligned}$$

where

- Virtual work is denoted δW and is zero for a body in equilibrium;
- External virtual work is δW_E, and;
- Internal virtual work is δW_I.

And so the external virtual work must equal the internal virtual work. It is in this form that the Principle of Virtual Work finds most use.

Application of Virtual Displacements

For a virtual displacement we have:

$$\delta W = 0$$
$$\delta W_E = \delta W_I$$
$$\sum F_i \cdot \delta y_i = \sum P_i \cdot \delta e_i$$

In which, for the external virtual work, F_i represents an externally applied force (or moment) and δy_i its virtual displacement. And for the internal virtual work, P_i represents the internal force (or moment) in member *i* and δe_i its virtual deformation. The summations reflect the fact that all work done must be accounted for.

Remember in the above, each the displacements must be compatible and the forces must be in equilibrium, summarized as:

Set of forces in equilibrium

$$\sum F_i \cdot \delta y_i = \sum P_i \cdot \delta e_i$$

Set of compatible displacements

Application of Virtual Forces

When virtual forces are applied, we have:

$$\delta W = 0$$
$$\delta W_E = \delta W_I$$
$$\sum y_i \cdot \delta F_i = \sum e_i \cdot \delta P_i$$

And again note that we have an equilibrium set of forces and a compatible set of displacements:

Set of compatible displacements

$$\sum y_i \cdot \delta F_i = \sum e_i \cdot \delta P_i$$

Set of forces in equilibrium

In this case the displacements are the real displacements that occur when the structure is in equilibrium and the virtual forces are any set of arbitrary forces that are in equilibrium.

2.2.2 Virtual Work for Deflections

Deflections in Beams and Frames

For a beam we proceed as:

1. Write the virtual work equation for bending:

$$\begin{aligned}\delta W &= 0\\ \delta W_E &= \delta W_I\\ y \cdot \delta F &= \sum \theta_i \cdot \delta M_i\end{aligned}$$

2. Place a unit load, δF, at the point at which deflection is required;
3. Find the real bending moment diagram, M_x, since the real curvatures are given by:

$$\theta_x = \frac{M_x}{EI_x}$$

4. Solve for the virtual bending moment diagram (the virtual force equilibrium set), δM, caused by the virtual unit load.
5. Solve the virtual work equation:

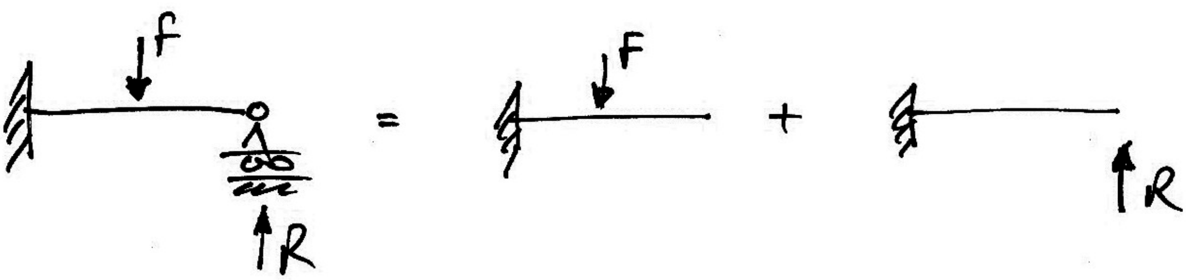
$$y \cdot 1 = \int_0^L \left[\frac{M_x}{EI}\right] \cdot \delta M_x \, dx$$

6. Note that the integration tables can be used for this step.

2.2.3 Virtual Work for Indeterminate Structures

General Approach

Using compatibility of displacement, we have:



Final = **Primary** + **Reactant**

Next, further break up the reactant structure, using linear superposition:

Reactant = **Multiplier** × **Unit Reactant**

We summarize this process as:

$$M = M^0 + \alpha M^1$$

- M is the force system in the original structure (in this case moments);
- M^0 is the primary structure force system;
- M^1 is the unit reactant structure force system.

The primary structure can be analysed, as can the unit reactant structure. Thus, the only unknown is the multiplier, α, for which we use virtual work to calculate.

Finding the Multiplier

For beams and frames, we have:

$$0 = \sum \int_0^L \frac{M^0 \cdot \delta M_i^1}{EI_i} dx + \alpha \cdot \sum \int_0^L \frac{\left(\delta M_i^1\right)^2}{EI_i} dx$$

Thus:

$$\alpha = \frac{-\sum \int_0^L \frac{M^0 \cdot \delta M_i^1}{EI_i} dx}{\sum \int_0^L \frac{\left(\delta M_i^1\right)^2}{EI_i} dx}$$

2.2.4 Virtual Work for Compound Structures

Basis

In the general equation for Virtual Work:

$$\sum y_i \cdot \delta F_i = \sum e_i \cdot \delta P_i$$

We note that the summation on the right hand side is over all forms of real displacement and virtual force combinations. For example, if a member is in combined bending and axial force, then we must include the work done by both effects:

$$\begin{aligned}\left(\delta W_i\right)_{\text{Member}} &= \left(e \cdot \delta P\right)_{\text{Axial}} + \left(e \cdot \delta P\right)_{\text{Bending}} \\ &= \frac{PL}{EA} \cdot \delta P + \int \frac{M}{EI} \cdot \delta M \; dx\end{aligned}$$

The total Virtual Work done by any member is:

$$\left(\delta W_i\right)_{\text{Member}} = \frac{PL}{EA} \cdot \delta P + \int \frac{M}{EI} \cdot \delta M \; dx + \frac{T}{GJ} \cdot \delta T + \frac{V}{GA_v} \cdot \delta V$$

In which Virtual Work done by axial, bending, torsion, and shear, respectively, is accounted for. However, most members primarily act through only one of these stress resultants, and so we commonly have only one term per member. A typical example is when axial deformation of frame (bending) members is neglected; since the area is large the contribution to virtual work is small.

At the level of the structure as a whole, we must account for all such sources of Virtual Work. For the typical structures we study here, we account for the Virtual Work done by axial and flexural members separately:

$$\begin{aligned} \delta W &= 0 \\ \delta W_E &= \delta W_I \\ \sum y_i \cdot \delta F_i &= \sum e_i \cdot \delta P_i + \sum \theta_i \cdot \delta M_i \end{aligned}$$

In which the first term on the RHS is the internal virtual work done by axial members and the second term is that done by flexural members.

Again considering only axial and bending members, if a deflection is sought:

$$\begin{aligned} y \cdot \delta F &= \sum e_i \cdot \delta P_i + \sum \theta_i \cdot \delta M_i \\ y \cdot 1 &= \sum \left(\frac{PL}{EA} \right)_i \cdot \delta P_i + \sum \int_0^L \left[\frac{M_x}{EI} \right] \cdot \delta M_x \, dx \end{aligned}$$

To solve such an indeterminate structure, we have the contributions to Virtual Work:

$$\begin{aligned} M &= M^0 + \alpha M^1 \\ P &= P^0 + \alpha P^1 \end{aligned}$$

for the structure as a whole. Hence we have:

$$\delta W = 0$$

$$\delta W_E = \delta W_I$$

$$\sum y_i \cdot \delta F_i = \sum e_i \cdot \delta P_i + \sum \theta_i \cdot \delta M_i$$

$$0 \cdot 1 = \sum \left(\frac{PL}{EA} \right)_i \cdot \delta P_i^1 + \sum \int_0^L \left[\frac{M_x}{EI} \right] \cdot \delta M_x \, dx$$

$$0 = \sum \left(\frac{\left(P^0 + \alpha \cdot \delta P^1\right) L}{EA} \right)_i \cdot \delta P_i^1 + \sum \int_0^L \left[\frac{\left(M_x^0 + \alpha M_x^1\right)}{EI} \right] \cdot \delta M_x \, dx$$

$$0 = \sum \left(\frac{P^0 L}{EA} \right)_i \cdot \delta P_i^1 + \alpha \cdot \sum \left(\frac{\delta P^1 L}{EA} \right)_i \cdot \delta P_i^1 + \sum \int_0^L \frac{M_x^0 \cdot \delta M_x^1}{EI} dx + \alpha \cdot \sum \int_0^L \frac{\left(\delta M_x^1\right)^2}{EI} dx$$

Hence the multiplier can be found as:

$$\alpha = -\frac{\sum \frac{P^0 \cdot \delta P_i^1 \cdot L_i}{EA_i} + \sum \int_0^L \frac{M^0 \cdot \delta M_i^1}{EI_i} dx}{\sum \frac{\left(\delta P_i^1\right)^2 L_i}{EA_i} + \sum \int_0^L \frac{\left(\delta M_i^1\right)^2}{EI_i} dx}$$

Note the negative sign!

Though these expressions are cumbersome, the ideas and the algebra are both simple.

Integration of Diagrams

We are often faced with the integration of various diagrams when using virtual work to calculate the deflections, etc. As such diagrams only have a limited number of shapes, a table of ‘volume’ integrals is used.

2.3 Basic Examples

2.3.1 Example 1

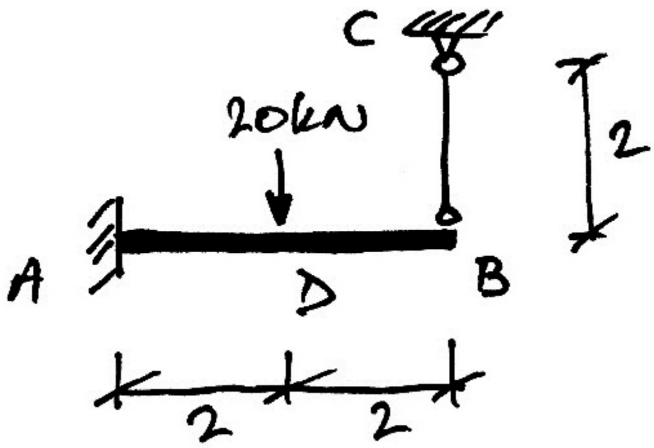
Problem

For the following structure, find:

(a) The force in the cable BC and the bending moment diagram;

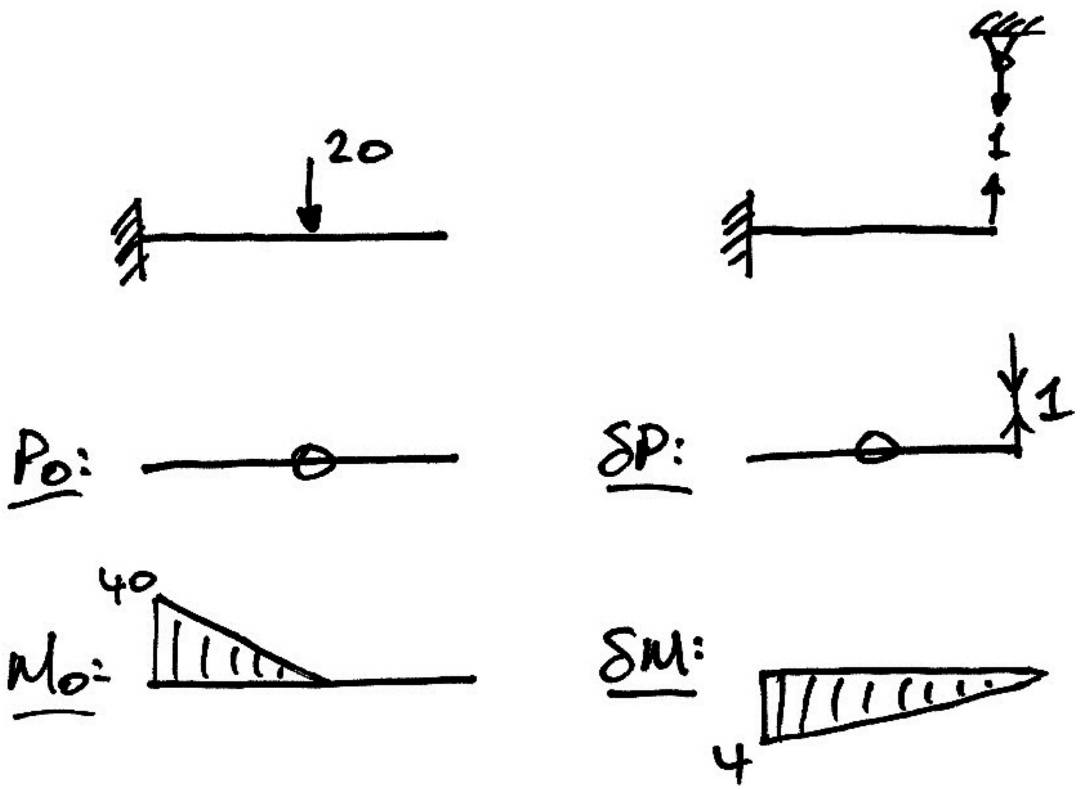
(b) The vertical deflection at D.

Take $EI = 8\times10^3 \text{kNm}^2$ and $EA = 16\times10^3$ kN.



Solution – Part (a)

This is a one degree indeterminate structure and so we must release one redundant. We could choose many, but the most obvious is the cable, *BC*. We next analyze the primary structure for the actual loads, and the unit virtual force placed in lieu of the redundant:



From the derivation of Virtual Work for indeterminate structures, we have:

$$0=\sum\left(\frac{P^0L}{EA}\right)_i\cdot\delta P_i^1+\alpha\cdot\sum\left(\frac{\delta P^1L}{EA}\right)_i\cdot\delta P_i^1+\sum\int_0^L\frac{M_x^0\cdot\delta M_x^1}{EI}dx+\alpha\cdot\sum\int_0^L\frac{\left(\delta M_x^1\right)^2}{EI}dx$$

We evaluate each term separately to simplify the calculations and to minimize potential calculation error.

Term 1:

This term is zero since P^0 is zero.

Term 2:

Only member *BC* contributes to this term and so it is:

$$\sum\left(\frac{\delta P^1 L}{EA}\right)_i \cdot \delta P_i^1 = \frac{1\cdot 2}{EA}\cdot 1 = \frac{2}{EA}$$

Term 3:

Here we must integrate the bending moment diagrams. We use the volume integral for the portion *AD* of both diagrams. Thus we multiply a triangle by a trapezoid:

$$\sum\int_0^L \frac{M_x^0 \cdot \delta M_x^1}{EI}dx = \frac{1}{EI}\left[\frac{1}{6}(40)\left(-2+2(-4)\right)(2)\right]$$
$$= -\frac{400/3}{EI}$$

Term 4:

Here we multiply the virtual BMD by itself so it is a triangle by a triangle:

$$\sum\int_0^L \frac{\left(\delta M_x^1\right)^2}{EI}dx = \frac{1}{EI}\left[\frac{1}{3}(-4)(-4)(4)\right] = \frac{64/3}{EI}$$

With all terms evaluated the Virtual Work equation becomes:

$$0 = 0 + \alpha\cdot\frac{2}{EA} - \frac{400/3}{EI} + \alpha\cdot\frac{64/3}{EI}$$

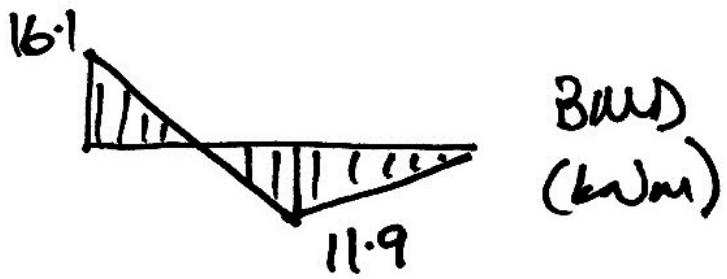
Which gives:

$$\alpha = \frac{\dfrac{400/3}{EI}}{\dfrac{2}{EA} + \dfrac{64/3}{EI}} = \frac{400}{6\dfrac{EI}{EA} + 64}$$

Given that $EI/EA = 8 \times 10^3 / 16 \times 10^3 = 0.5$, we have:

$$\alpha = \frac{400}{6(0.5) + 64} = 5.97$$

Thus there is a tension (positive answer) in the cable of 5.97 kN, giving the BMD as:

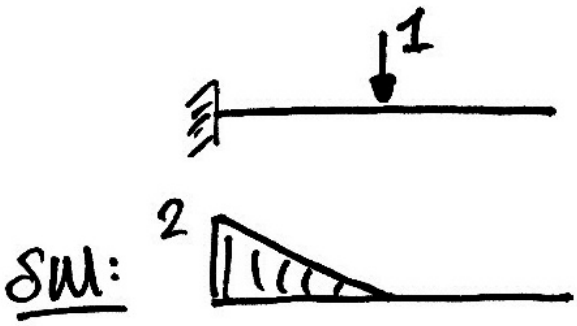


Note that this comes from:

$$M_A = M^0 + \alpha \cdot \delta M = 40 + (5.97)(-4) = 16.1 \text{ kN}$$
$$M_D = M^0 + \alpha \cdot \delta M = 0 + (5.97)(-2) = -11.9 \text{ kN}$$

Solution – Part (b)

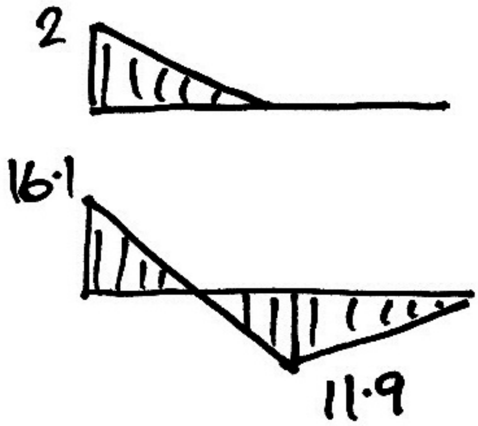
Recalling that the only requirement on applying virtual forces to calculate real displacements is that an equilibrium system results, we can apply a vertical unit force at *D* to the primary structure only:



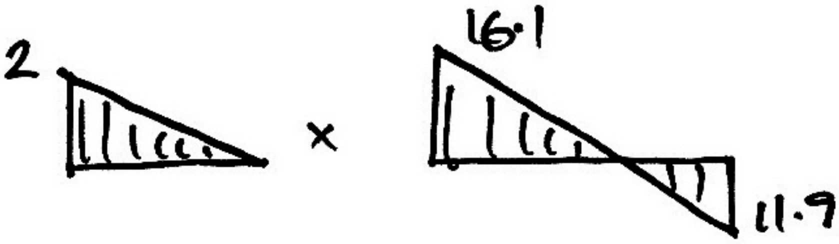
The Virtual Work equation useful for deflection is:

$$y \cdot \delta F = \sum e_i \cdot \delta P_i + \sum \theta_i \cdot \delta M_i$$
$$\delta_{Dy} \cdot 1 = \sum \left(\frac{PL}{EA} \right)_i \cdot \delta P_i + \sum \int_0^L \left[\frac{M_x}{EI} \right] \cdot \delta M_x \, dx$$

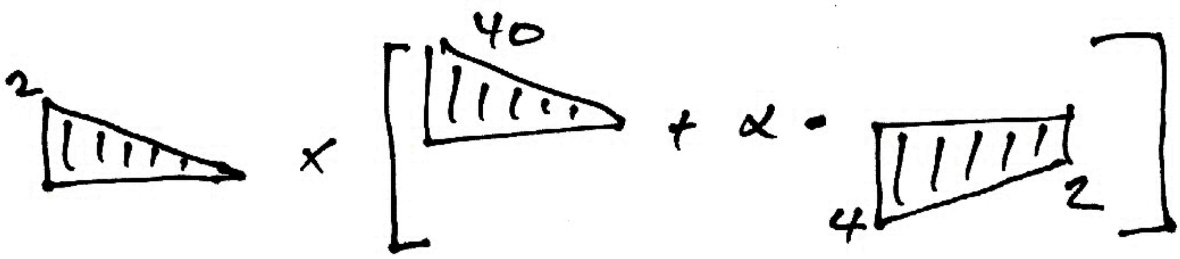
Since $\delta P = 0$, we need only calculate the term involving the Virtual Work done by the beam bending. This involves the volume integral of the two diagrams:



Note that only the portion *AD* will count as there is no virtual moment on *DB*. Thus we have:



However, this shape is not easy to work with, given the table to hand. Therefore we recall that the real BMD came about as the superposition of two BMD shapes that are easier to work with, and so we have:



A further benefit of this approach is that an equation of deflection in terms of the multiplier α is got. This could then be used to determine α for a particular design requirement, and in turn this could inform the choice of EI/EA ratio. Thus:

$$
\begin{aligned}
\delta_{Dy} &= \sum \int_0^L \left[\frac{M_x}{EI} \right] \cdot \delta M_x \, dx \\
&= \frac{1}{EI} \left[\frac{1}{3}(2)(40)(2) + \alpha \cdot \frac{1}{6}(2)\left(-2+2(-4)\right)(2) \right] \\
&= \frac{160 - 20\alpha}{3EI}
\end{aligned}
$$

Given $\alpha = 5.97$, we then have:

$$\delta_{Dy} = \frac{160 - 20(5.97)}{3EI} = \frac{13.9}{EI} = \frac{13.9}{8 \times 10^3} \times 10^3 = 1.7 \text{ mm}$$

The positive answer indicates that the deflection is in the direction of the applied virtual vertical force and so is downwards as expected.

We can also easily work out the deflection at *B*, since it is the same as the elongation of the cable:

$$\delta_{By} = \frac{PL}{EA} = \frac{(5.97)(2)}{16 \times 10^3} \times 10^3 = 0.75 \text{ mm}$$

Draw the deflected shape of the structure.

2.3.2 Example 2

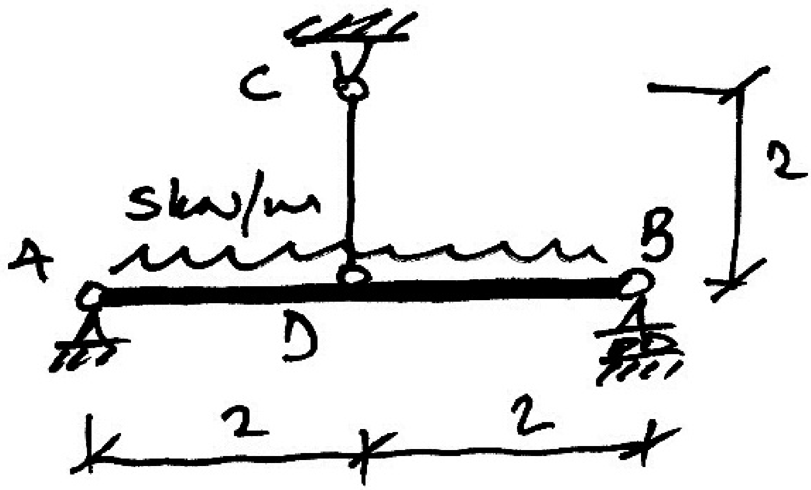
Problem

For the following structure, find:

(a) The force in the cable *CD* and the bending moment diagram;

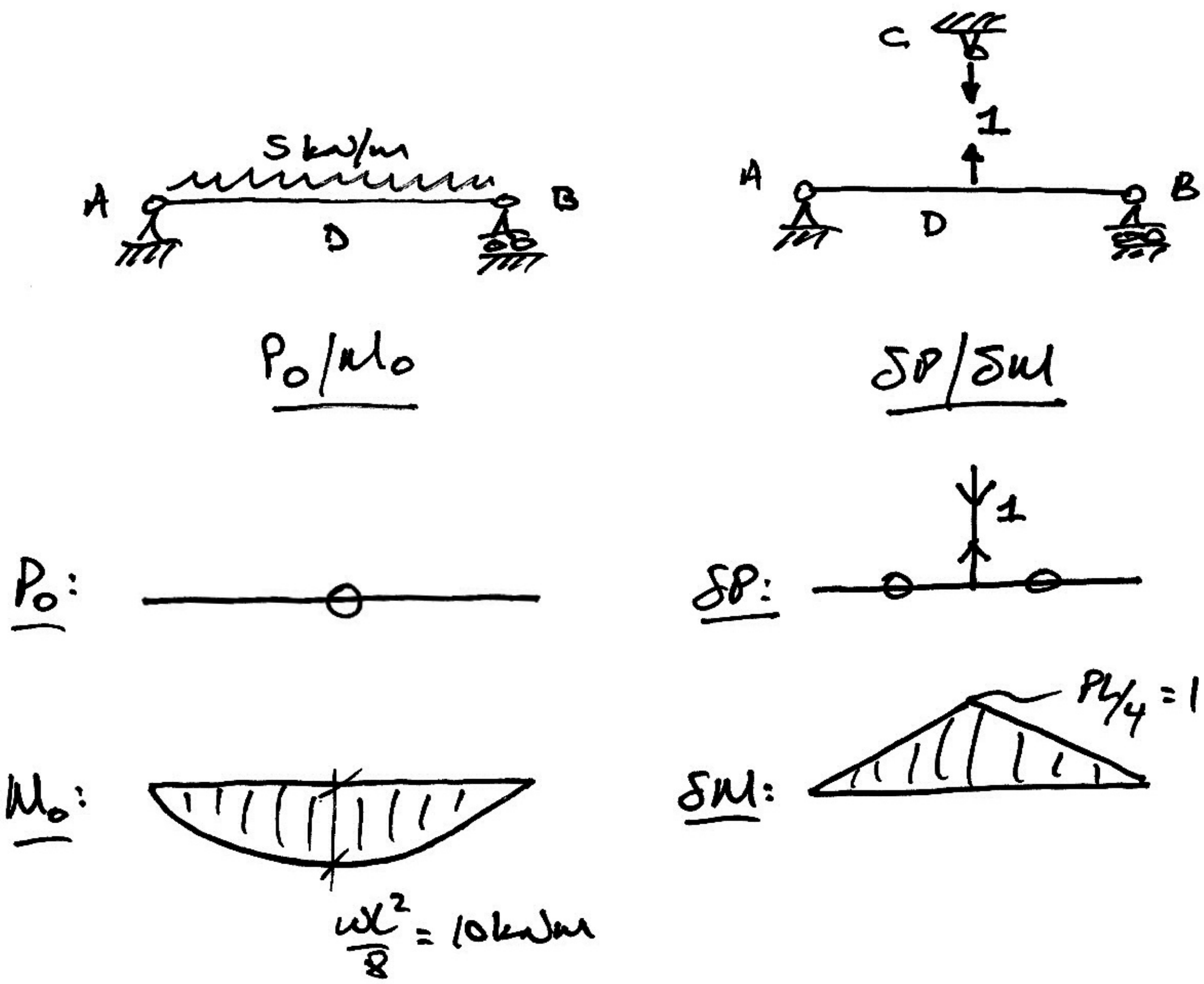
(b) Determine the optimum *EA* of the cable for maximum efficiency of the beam.

Take $EI = 8\times10^3 \text{ kNm}^2$ and $EA = 48\times10^3 \text{ kN}$.



Solution – Part (a)

Choose the cable *CD* as the redundant to give:



The equation of Virtual Work relevant is:

$$0 = \sum \left(\frac{P^0 L}{EA} \right)_i \cdot \delta P_i^1 + \alpha \cdot \sum \left(\frac{\delta P^1 L}{EA} \right)_i \cdot \delta P_i^1 + \sum \int_0^L \frac{M_x^0 \cdot \delta M_x^1}{EI} dx + \alpha \cdot \sum \int_0^L \frac{\left(\delta M_x^1 \right)^2}{EI} dx$$

We evaluate each term separately:

Term 1:

This term is zero since P^0 is zero.

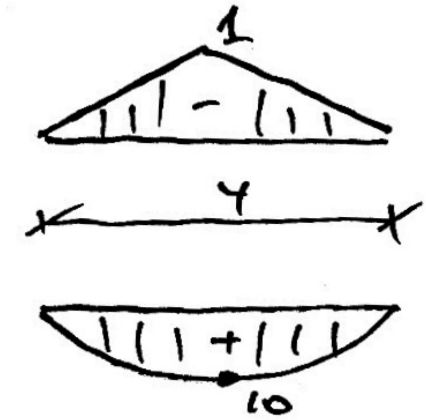
Term 2:

Only member *CD* contributes to this term and so it is:

$$\sum\left(\frac{\delta P^1 L}{EA}\right)_i \cdot \delta P_i^1 = \frac{1\cdot 2}{EA}\cdot 1 = \frac{2}{EA}$$

Term 3:

Here we must integrate the bending moment diagrams. We use the volume integral for each half of the diagram, and multiply by 2, since we have two such halves.



$$\sum\int_0^L \frac{M_x^0 \cdot \delta M_x^1}{EI}dx = \frac{2}{EI}\left[\frac{5}{12}(-1)(10)(2)\right]$$
$$= -\frac{50/3}{EI}$$

Term 4:

Here we multiply the virtual BMD by itself:

$$\sum\int_0^L \frac{\left(\delta M_x^1\right)^2}{EI}dx = \frac{2}{EI}\left[\frac{1}{3}(-1)(-1)(2)\right] = \frac{4/3}{EI}$$

Thus the Virtual Work equation becomes:

$$0 = 0 + \alpha \cdot \frac{2}{EA} - \frac{50/3}{EI} + \alpha \cdot \frac{4/3}{EI}$$

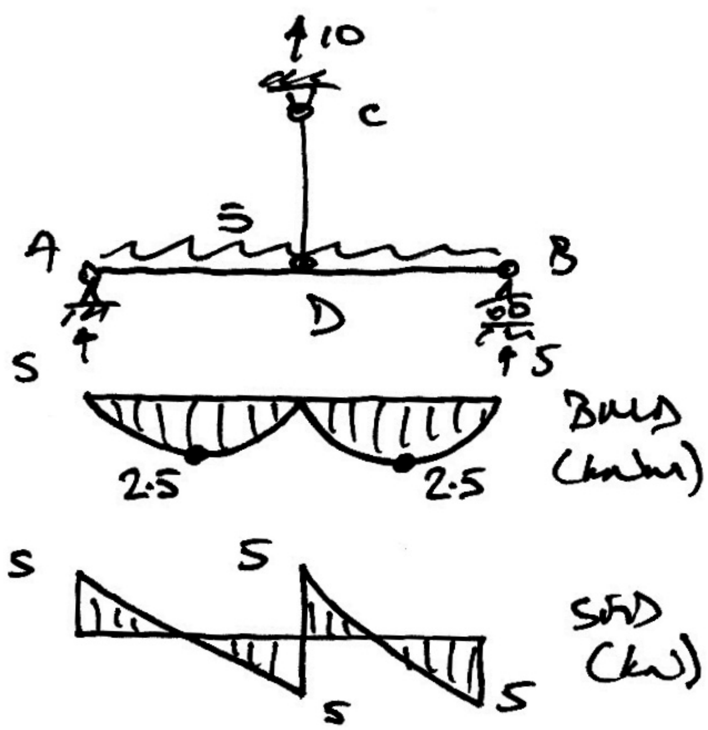
Which gives:

$$\alpha = \frac{\frac{50/3}{EI}}{\frac{2}{EA} + \frac{4/3}{EI}} = \frac{50}{6\frac{EI}{EA} + 4}$$

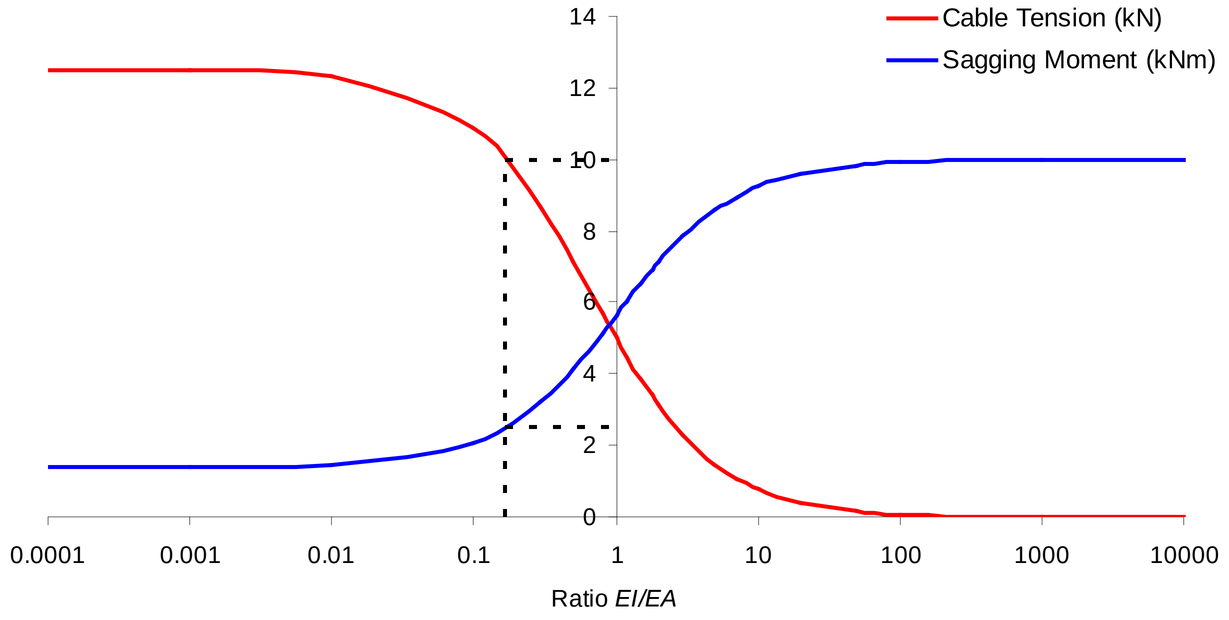
Given that $EI/EA = 8 \times 10^3 / 48 \times 10^3 = 0.167$, we have:

$$\alpha = \frac{50}{6(0.167) + 4} = 10$$

Thus there is a tension (positive answer) in the cable of 10 kN, giving:

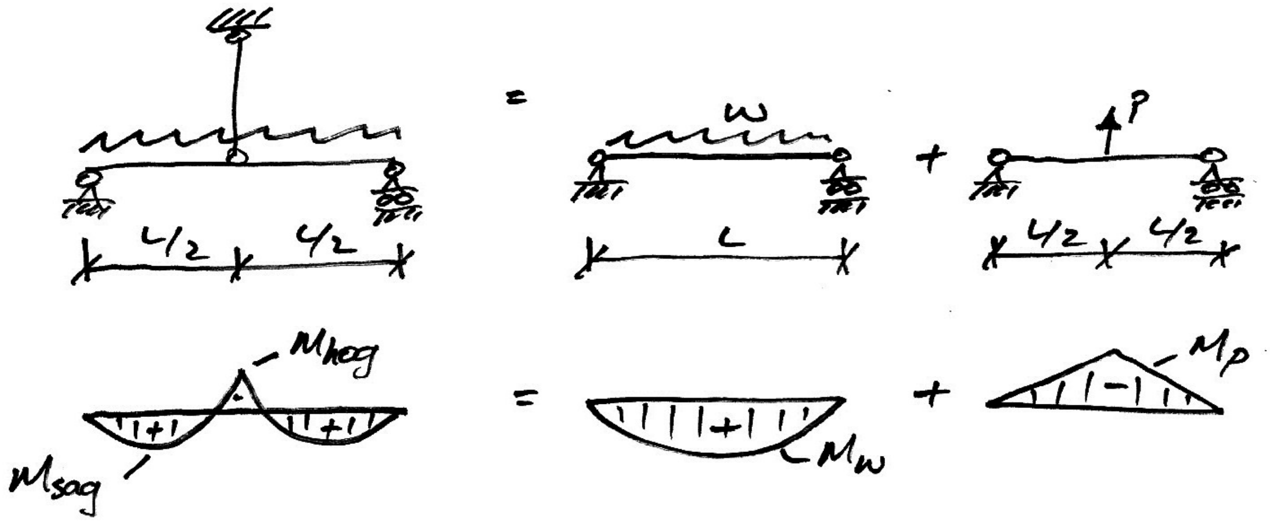


As designers, we want to control the flow of forces. In this example we can see that by changing the ratio EI/EA we can control the force in the cable, and the resulting bending moments. We can plot the cable force and maximum sagging bending moment against the stiffness ratio to see the behaviour for different relative stiffnesses:

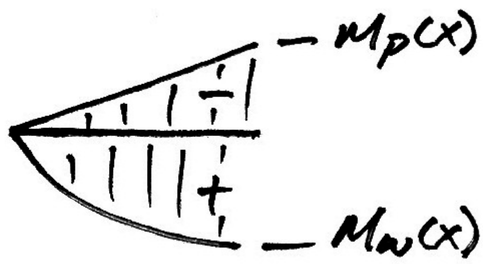


Solution – Part (b)

Efficiency of the beam means that the moments are resisted by the smallest possible beam. Thus the largest moment anywhere in the beam must be made as small as possible. Therefore the hogging and sagging moments should be equal:



We know that the largest hogging moment will occur at $L/2$. However, we do not know where the largest sagging moment will occur. Lastly, we will consider sagging moments positive and hogging moments negative. Consider the portion of the net bending moment diagram, $M(x)$, from 0 to $L/2$:



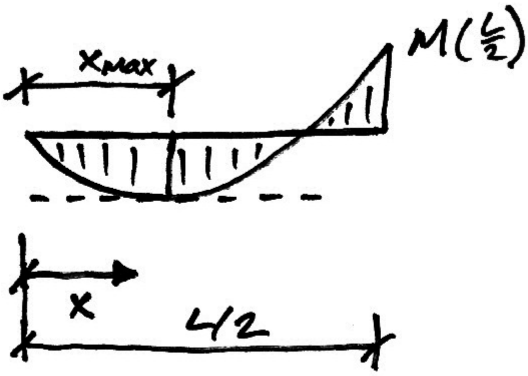
The equations of these bending moments are:

$$M_P(x) = -\frac{P}{2}x$$

$$M_W(x) = -\frac{w}{2}x^2 + \frac{wL}{2}x$$

Thus:

$$\begin{aligned} M(x) &= M_W(x) + M_P(x) \\ &= \frac{wL}{2}x - \frac{w}{2}x^2 - \frac{P}{2}x \end{aligned}$$



The moment at $L/2$ is:

$$\begin{aligned} M(L/2) &= \frac{wL}{2}\left(\frac{L}{2}\right) - \frac{w}{2}\left(\frac{L}{2}\right)^2 - \frac{P}{2}\left(\frac{L}{2}\right) \\ &= \frac{wL^2}{4} - \frac{wL^2}{8} - \frac{PL}{4} \\ &= \frac{wL^2}{8} - \frac{PL}{4} \end{aligned}$$

Which is as we expected. The maximum sagging moment between 0 and $L/2$ is found at:

$$\frac{dM(x)}{dx} = 0$$
$$\frac{wL}{2} - wx_{\max} - \frac{P}{2} = 0$$
$$x_{\max} = \frac{L}{2} - \frac{P}{2w}$$

Thus the maximum sagging moment has a value:

$$\begin{aligned} M(x_{\max}) &= \frac{wL}{2}\left(\frac{L}{2} - \frac{P}{2w}\right) - \frac{w}{2}\left(\frac{L}{2} - \frac{P}{2w}\right)^2 - \frac{P}{2}\left(\frac{L}{2} - \frac{P}{2w}\right) \\ &= \frac{wL^2}{4} - \frac{PL}{4} - \frac{w}{2}\left(\frac{L^2}{4} - \frac{2PL}{4w} + \frac{P^2}{4w}\right) - \frac{PL}{4} + \frac{P^2}{4w} \\ &= \frac{wL^2}{8} - \frac{PL}{4} + \frac{P^2}{8w} \end{aligned}$$

Since we have assigned a sign convention, the sum of the hogging and sagging moments should be zero, if we are to achieve the optimum BMD. Thus:

$$M(x_{\max}) + M(L/2) = 0$$
$$\left[\frac{wL^2}{8} - \frac{PL}{4} + \frac{P^2}{8w}\right] + \left[\frac{wL^2}{8} - \frac{PL}{4}\right] = 0$$
$$\frac{wL^2}{4} - \frac{PL}{2} + \frac{P^2}{8w} = 0$$
$$\left(\frac{1}{8w}\right)P^2 + \left(-\frac{L}{2}\right)P + \left(\frac{wL^2}{4}\right) = 0$$

This is a quadratic equation in P and so we solve for P using the usual method:

$$P = \frac{\frac{L}{2} \pm \sqrt{\frac{L^2}{4} - \frac{L^2}{8}}}{\frac{2}{8w}} = \frac{8w}{2}\left(\frac{L}{2} \pm \frac{L}{\sqrt{8}}\right)$$
$$= wL\left(2 \pm \sqrt{2}\right)$$

Since the load in the cable must be less than the total amount of load in the beam, that is, $P < wL$, we have:

$$P = wL\left(2 - \sqrt{2}\right) = 0.586wL$$

With this value for *P* we can determine the hogging and sagging moments:

$$\begin{aligned} M\left(L/2\right) &= \frac{wL^2}{8} - \frac{wL\left(2-\sqrt{2}\right)L}{4} \\ &= wL^2\left(\frac{2\sqrt{2}-3}{8}\right) \\ &= -0.0214wL^2 \end{aligned}$$

And:

$$\begin{aligned} M\left(x_{\max}\right) &= \left(\frac{wL^2}{8} - \frac{PL}{4}\right) + \frac{P^2}{8w} \\ &= wL^2\left(\frac{2\sqrt{2}-3}{8}\right) + \frac{\left[wL\left(2-\sqrt{2}\right)\right]^2}{8w} \\ &= wL^2\left(\frac{3-2\sqrt{2}}{8}\right) \\ &= +0.0214wL^2 \end{aligned}$$

Lastly, the location of the maximum sagging moment is given by:

$$\begin{aligned} x_{\max} &= \frac{L}{2} - \frac{P}{2w} \\ &= \frac{L}{2} - \frac{wL\left(2-\sqrt{2}\right)}{2w} \\ &= \frac{L}{2}\left(\sqrt{2}-1\right) \\ &= 0.207L \end{aligned}$$

For our particular problem, $w = 5$ kN/m, $L = 4$ m, giving:

$$P = 0.586\left(5 \times 4\right) = 11.72 \text{ kN}$$

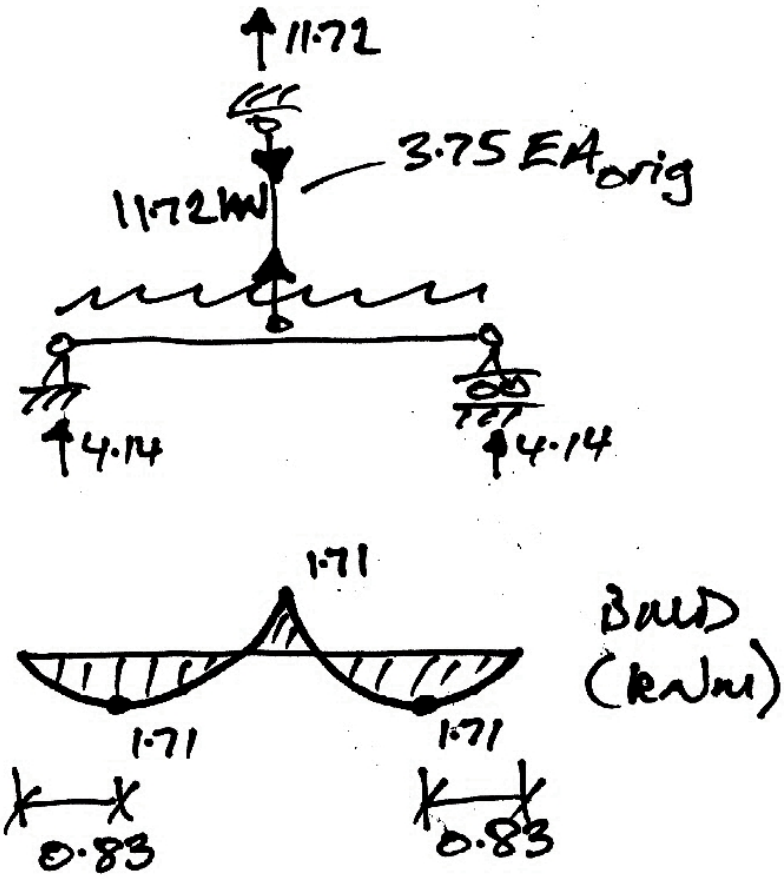
$$M\left(x_{\max}\right) = 0.0214\left(5 \times 4^2\right) = 1.71 \text{ kNm}$$

Thus, as we expected, $P > 10$ kN, the value obtained from Part (a) of the problem.

Now since, we know P we now also know the required value of the multiplier, α. Hence, we write the virtual work equations again, but this time keeping Term 2 in terms of *L*, since that is what we wish to solve for:

$$\begin{aligned} \alpha &= \frac{50}{6\dfrac{EI}{EA} + 4} = 11.72 \\ \therefore \frac{EI}{EA} &= \frac{1}{6}\left(\frac{50}{11.72} - 4\right) = 0.044 \end{aligned}$$

Giving $EA = 8 \times 10^3 / 0.044 = 180.3 \times 10^3$ kN. This is 3.75 times the original cable area – a lot of extra material just to change the cable force by 17%. However, there is a large saving by reducing the overall moment in the beam from 10 kNm (simply-supported) or 2.5 kNm (two-span beam) to 1.71 kNm.



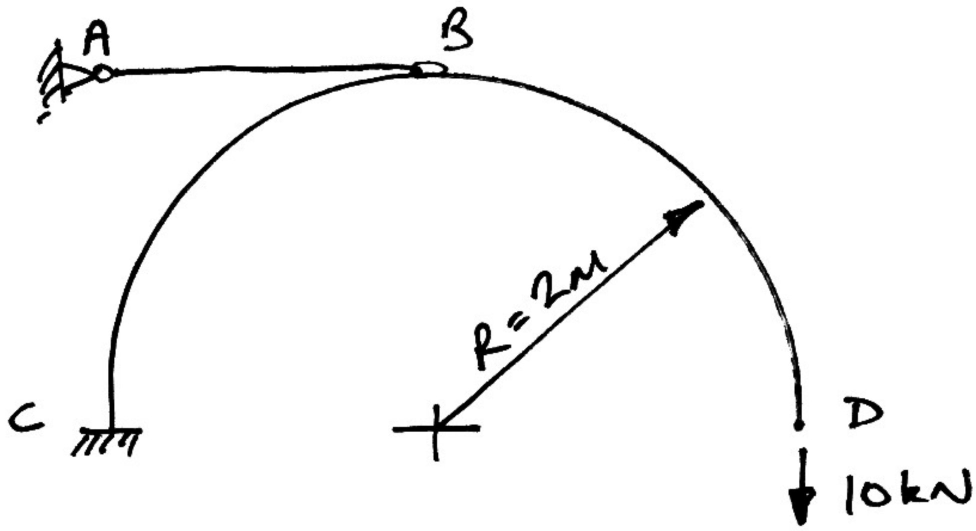
2.3.3 Example 3

Problem

For the following structure:

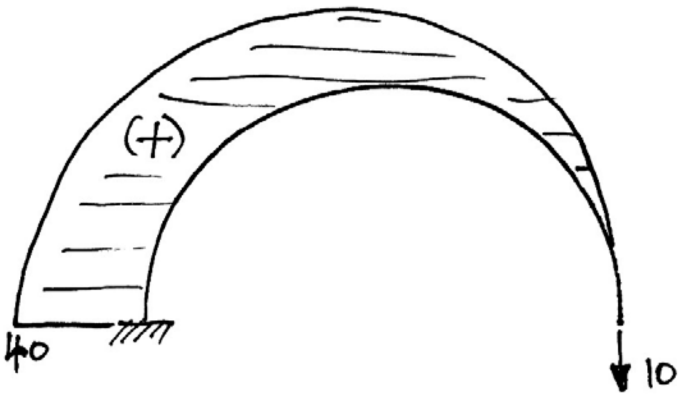
1. Determine the tension in the cable *AB*;
2. Draw the bending moment diagram;
3. Determine the vertical deflection at *D* with and without the cable *AB*.

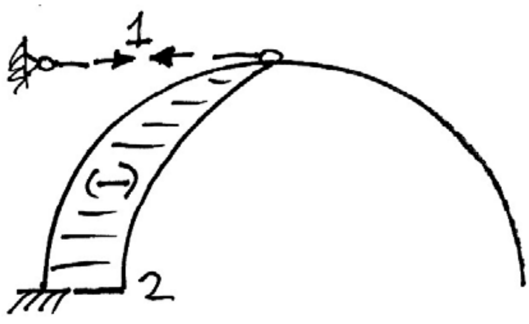
Take $EI = 120\times10^3$ kNm^2 and $EA = 60\times10^3$ kN.



Solution

As is usual, we choose the cable to be the redundant member and split the frame up as follows:



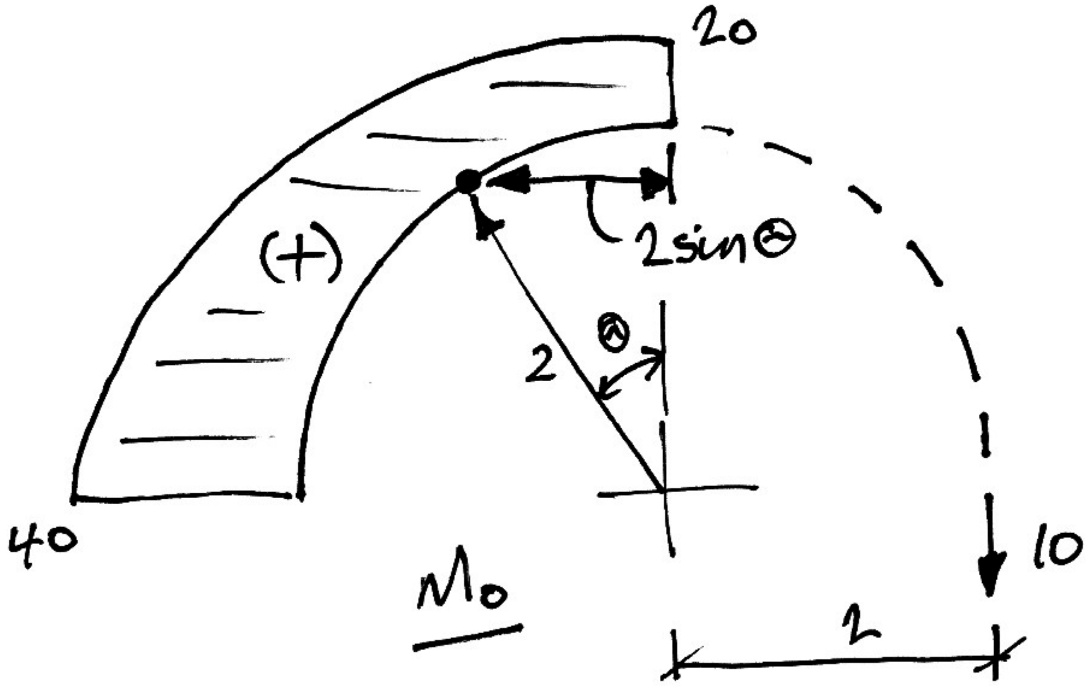


Primary Structure **Redundant Structure**

We must examine the BMDs carefully, and identify expressions for the moments around the arch. However, since we will be using virtual work and integrating one diagram against another, we immediately see that we are only interested in the portion of the structure *CB*. Further, we will use the anti-clockwise angle from vertical as the basis for our integration.

Primary BMD

Drawing the BMD and identify the relevant distances:

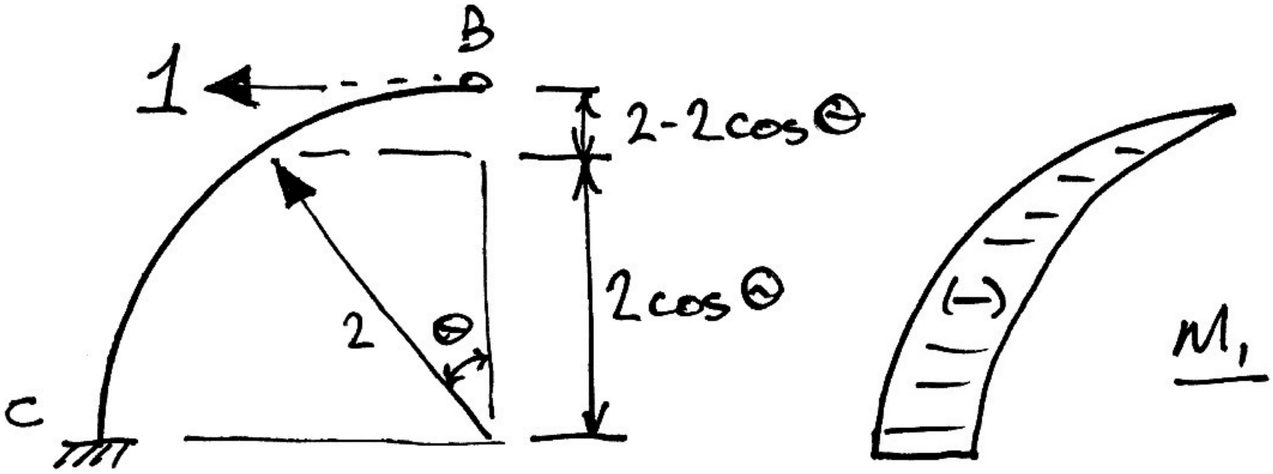


Hence the expression for M^0 is:

$$M_\theta^0 = 20 + 10\left(2\sin\theta\right) = 20\left(1 + \sin\theta\right)$$

Reactant BMD

This calculation is slightly easier:



$$M_{\theta}^{1} = 1 \cdot \left(2 - 2\cos\theta\right) = 2\left(1 - \cos\theta\right)$$

Virtual Work Equation

As before, we have the equation:

$$0 = \sum \left(\frac{P^0 L}{EA} \right)_i \cdot \delta P_i^1 + \alpha \cdot \sum \left(\frac{\delta P^1 L}{EA} \right)_i \cdot \delta P_i^1 + \sum \int_0^L \frac{M_x^0 \cdot \delta M_x^1}{EI} dx + \alpha \cdot \sum \int_0^L \frac{\left(\delta M_x^1\right)^2}{EI} dx$$

Term 1 is zero since there are no axial forces in the primary structure. We take each other term in turn.

Term 2

Since only member *AB* has axial force:

$$\text{Term 2} = \frac{\left(1\right)^2 2}{EA} = \frac{2}{EA}$$

Term 3

Since we want to integrate around the member – an integrand ds - but only have the moment expressed according to θ, we must change the integration limits by substituting:

$$ds = R \cdot d\theta = 2d\theta$$

Hence:

$$\begin{aligned}
\sum\int_0^L \frac{M_x^0 \cdot \delta M_x^1}{EI} dx &= \frac{1}{EI}\int_0^{\pi/2} \left[-2\left(1-\cos\theta\right)\right]\left[20\left(1+\sin\theta\right)\right] 2d\theta \\
&= \frac{80}{EI}\int_0^{\pi/2} \left(-1+\cos\theta\right)\left(1+\sin\theta\right) d\theta \\
&= \frac{80}{EI}\int_0^{\pi/2} \left(-1-\sin\theta+\cos\theta+\cos\theta\sin\theta\right) d\theta
\end{aligned}$$

To integrate this expression we refer to the appendix of integrals to get each of the terms, which then give:

$$\begin{aligned}
\sum\int_0^L \frac{M_x^0 \cdot \delta M_x^1}{EI} dx &= \frac{80}{EI}\left[-\theta+\cos\theta+\sin\theta-\frac{1}{4}\cos 2\theta\right]_0^{\pi/2} \\
&= \frac{80}{EI}\left\{\left[-\frac{\pi}{2}+0+1-\frac{1}{4}(-1)\right]-\left[-0+1+0-\frac{1}{4}\right]\right\} \\
&= \frac{80}{EI}\left(-\frac{\pi}{2}+1+\frac{1}{4}-1+\frac{1}{4}\right) \\
&= \frac{80}{EI}\left(\frac{1-\pi}{2}\right)
\end{aligned}$$

Term 4

Proceeding similarly to Term 3, we have:

$$\begin{aligned}\sum\int_0^L \frac{\left(\delta M_x^1\right)^2}{EI}dx &= \frac{1}{EI}\int_0^{\pi/2}\left[2\left(1-\cos\theta\right)\right]\left[2\left(1-\cos\theta\right)\right]2d\theta \\ &= \frac{8}{EI}\int_0^{\pi/2}\left(1-2\cos\theta+\cos^2\theta\right)d\theta\end{aligned}$$

Again we refer to the integrals appendix, and so for Term 4 we then have:

$$\begin{aligned}\sum\int_0^L \frac{\left(\delta M_x^1\right)^2}{EI}dx &= \frac{8}{EI}\int_0^{\pi/2}\left(1-2\cos\theta+\cos^2\theta\right)d\theta \\ &= \frac{8}{EI}\left[\theta-2\sin\theta+\left(\frac{\theta}{2}+\frac{1}{4}\sin 2\theta\right)\right]_0^{\pi/2} \\ &= \frac{8}{EI}\left\{\left[\frac{\pi}{2}-2+\left(\frac{\pi}{4}+\frac{1}{4}\right)\right]-\left[0-0+0+0\right]\right\} \\ &= \frac{8}{EI}\left(\frac{3\pi-7}{4}\right)\end{aligned}$$

Solution

Substituting the calculated values into the virtual work equation gives:

$$0 = 0+\alpha\cdot\frac{2}{EA}+\frac{80}{EI}\left(\frac{1-\pi}{2}\right)+\alpha\cdot\frac{8}{EI}\left(\frac{3\pi-7}{4}\right)$$

And so:

$$\alpha = \frac{-\frac{80}{EI}\left(\frac{1-\pi}{2}\right)}{\frac{2}{EA}+\frac{8}{EI}\left(\frac{3\pi-7}{4}\right)}$$

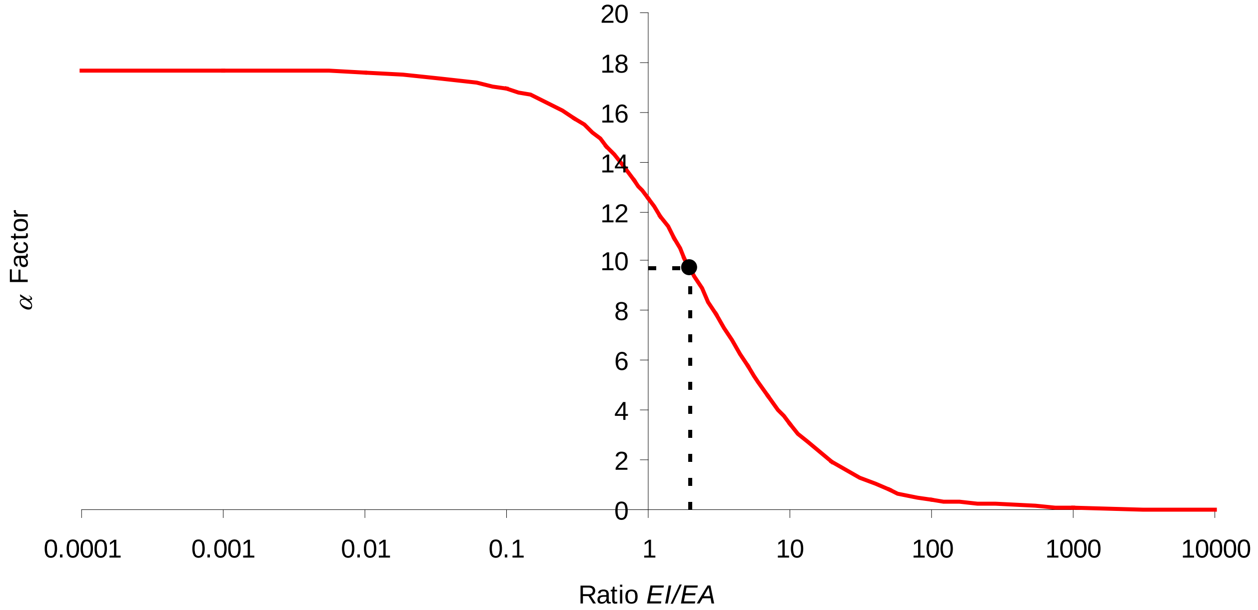
Simplifying:

$$\alpha = \frac{20\pi - 20}{3\pi - 7 + \frac{EI}{EA}}$$

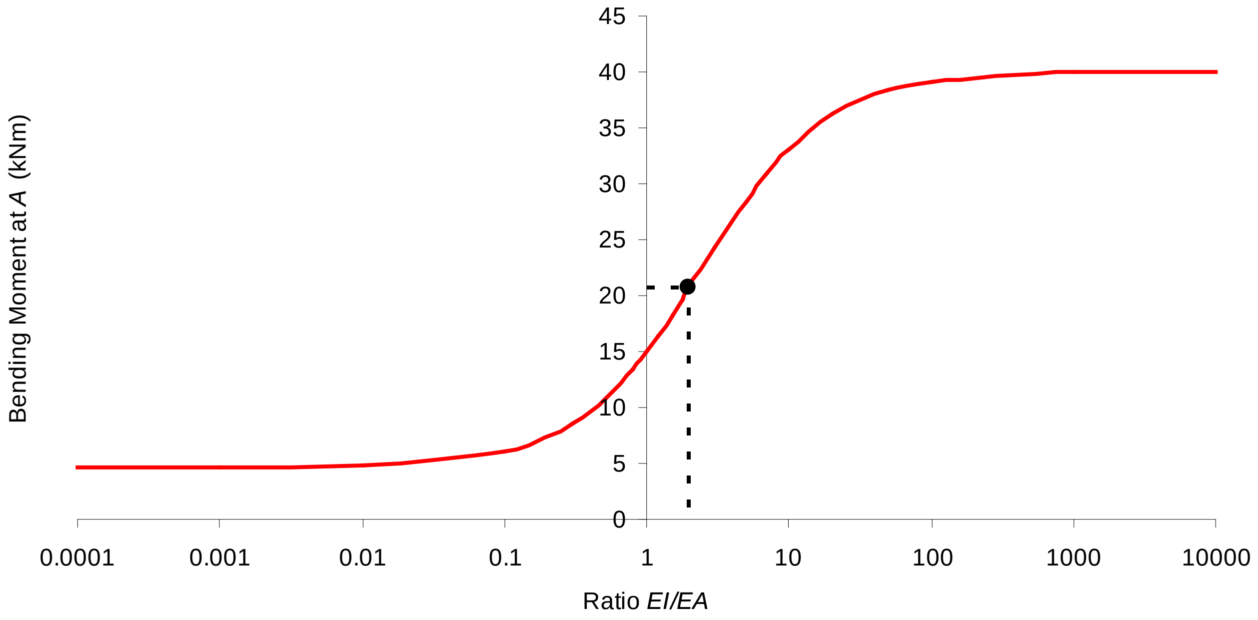
In this problem, $EI/EA = 2$ and so:

$$\alpha = \frac{20\pi - 20}{3\pi - 5} = 9.68 \text{ kN}$$

We can examine the effect of different ratios of EI/EA on the structure from our algebraic solution for α. We show this, as well as a point representing the solution for this particular EI/EA ratio on the following graph:



As can be seen, by choosing a stiffer frame member (increasing *EI*) or by reducing the area of the cable, we can reduce the force in the cable (which is just $1 \cdot \alpha$). However this will have the effect of increasing the moment at *A*, for example:



Deflections and shear would also be affected.

Draw the final BMD and determine the deflection at *D*.

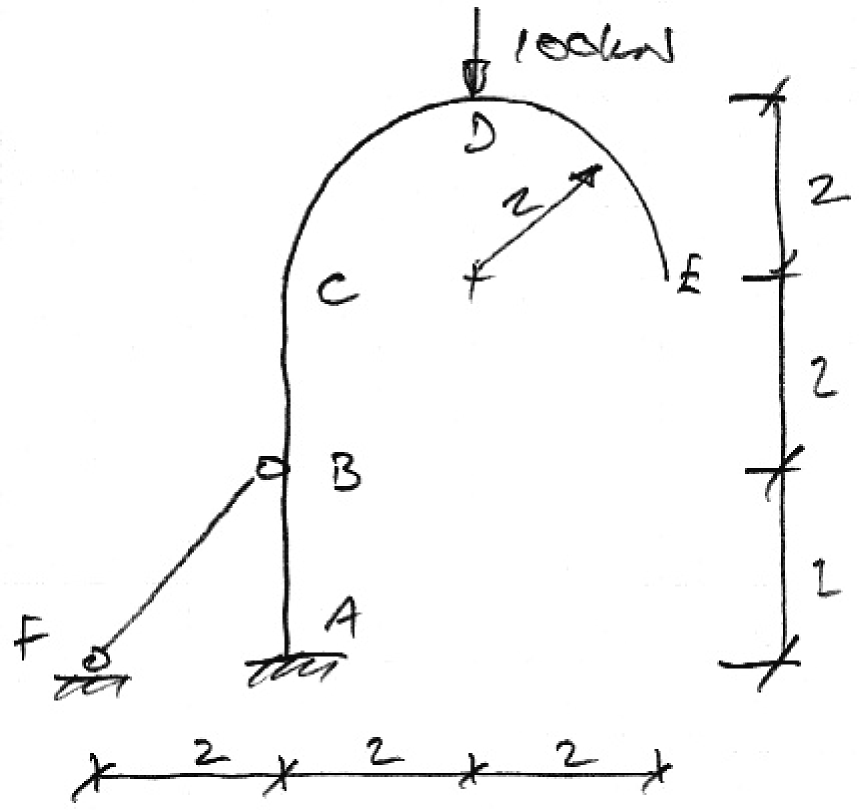
2.3.4 Example 4

Problem

For the following structure:

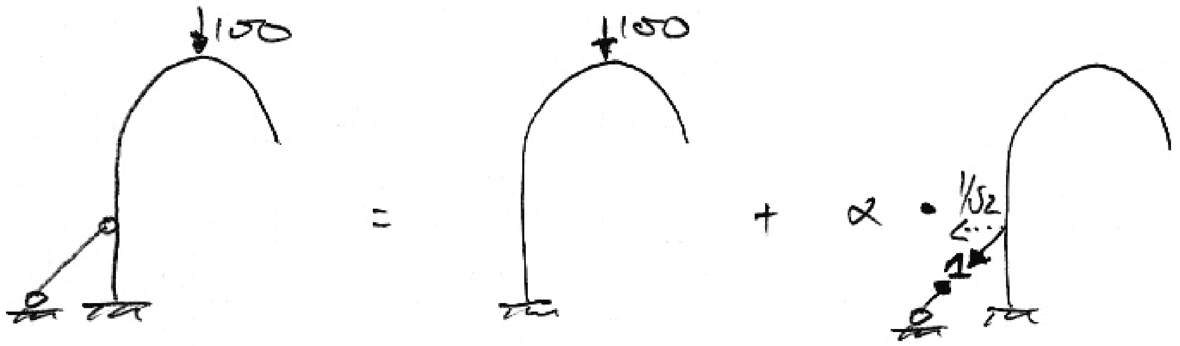
1. draw the bending moment diagram;
2. Find the vertical deflection at *E*.

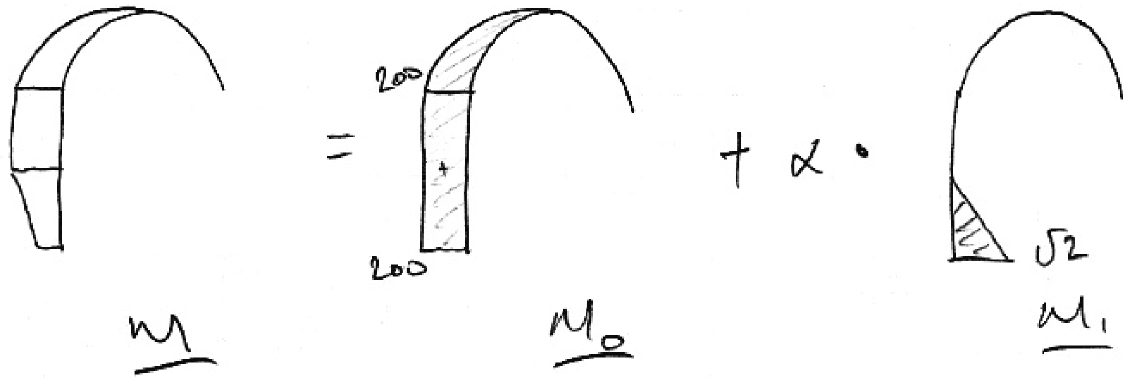
Take $EI = 120 \times 10^3$ kNm^2 and $EA = 60 \times 10^3$ kN .



Solution

To begin we choose the cable *BF* as the obvious redundant, yielding:





Virtual Work Equation

The Virtual Work equation is as before:

$$0 = \sum \left(\frac{P^0 L}{EA} \right)_i \cdot \delta P_i^1 + \alpha \cdot \sum \left(\frac{\delta P^1 L}{EA} \right)_i \cdot \delta P_i^1 + \sum \int_0^L \frac{M_x^0 \cdot \delta M_x^1}{EI} dx + \alpha \cdot \sum \int_0^L \frac{\left(\delta M_x^1 \right)^2}{EI} dx$$

Term 1 is zero since there are no axial forces in the primary structure. As we have done previously, we take each other term in turn.

Term 2

Though member *AB* has axial force, it is primarily a flexural member and so we only take account of the axial force in the cable *BF*:

$$\sum\left(\frac{\delta P^1 L}{EA}\right)_i \cdot \delta P_i^1 = \left(\frac{1\cdot 2\sqrt{2}}{EA}\right)\cdot 1 = \frac{2\sqrt{2}}{EA}$$

Term 3

Since only the portion *AB* has moment on both diagrams, it is the only section that requires integration here. Thus:

$$\sum\int_0^L \frac{M_x^0 \cdot \delta M_x^1}{EI}dx = \frac{1}{EI}\left[\frac{1}{2}(200)\left(-\sqrt{2}\right)(2)\right] = \frac{-220\sqrt{2}}{EI}$$

Term 3

Similar to Term 3, we have:

$$\sum\int_0^L \frac{\left(\delta M_x^1\right)^2}{EI}dx = \frac{1}{EI}\left[\frac{1}{3}\left(-\sqrt{2}\right)\left(-\sqrt{2}\right)(2)\right] = \frac{4/3}{EI}$$

Solution

Substituting the calculated values into the virtual work equation gives:

$$0 = 0 + \alpha\cdot\frac{2\sqrt{2}}{EA} - \frac{220\sqrt{2}}{EI} + \alpha\cdot\frac{4/3}{EI}$$

Thus:

$$\alpha = \frac{220\sqrt{2}\big/EI}{\dfrac{2\sqrt{2}}{EA} + \dfrac{4/3}{EI}}$$

And so:

$$\alpha = \frac{220\sqrt{2}}{2\sqrt{2}\dfrac{EI}{EA} + \dfrac{4}{3}}$$

Since:

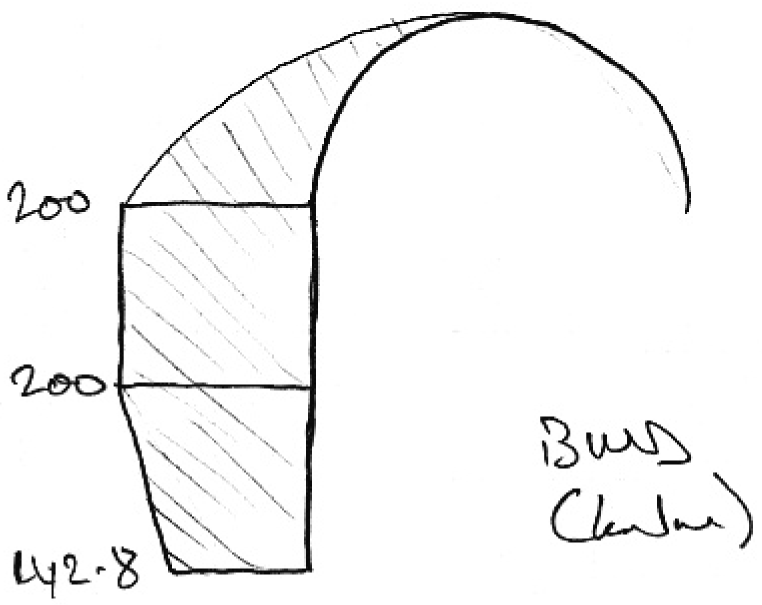
$$\frac{EI}{EA} = \frac{120 \times 10^3}{60 \times 10^3} = 2$$

We have:

$$\alpha = \frac{220\sqrt{2}}{\left(2\sqrt{2}\right)2 + \dfrac{4}{3}} = +40.46$$

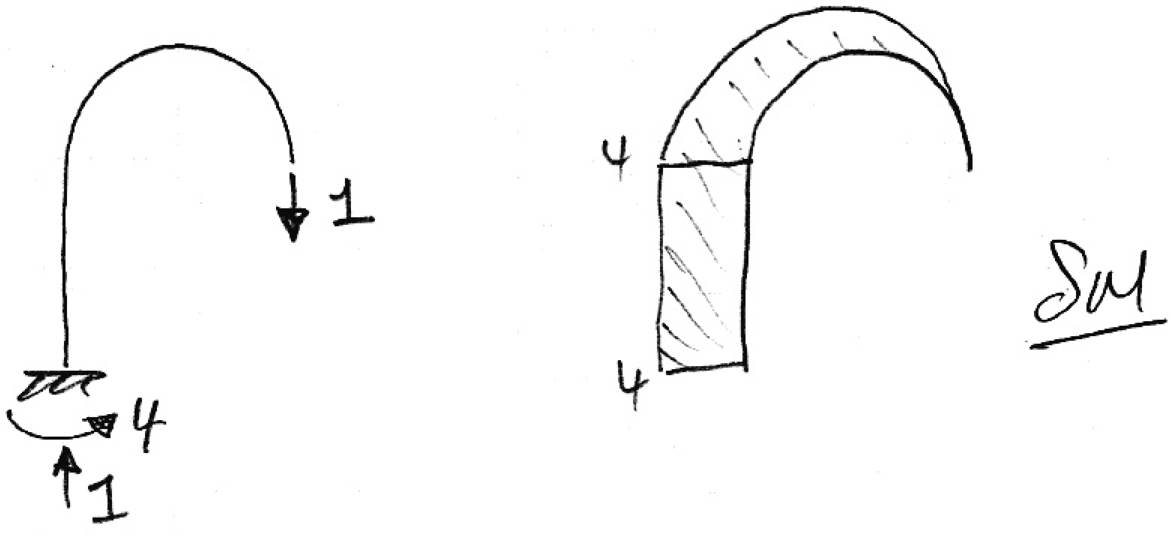
Thus the force in the cable *BF* is 40.46 kN tension, as assumed.

The bending moment diagram follows from superposition of the two previous diagrams:



To find the vertical deflection at *E*, we must apply a unit vertical load at *E*. We will apply a downwards load since we think the deflection is downwards. Therefore we should get a positive result to confirm our expectation.

We need not apply the unit vertical force to the whole structure, as it is sufficient to apply it to a statically determinate sub-structure. Thus we apply the force as follows:

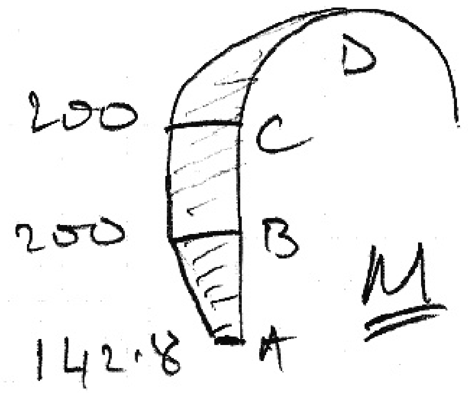


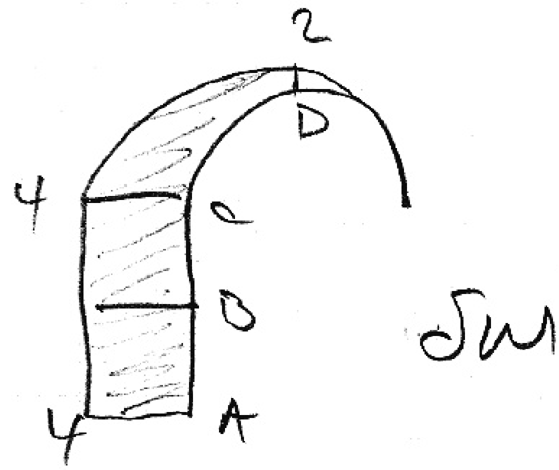
For the deflection, we have the following equation:

$$y \cdot \delta F = \sum e_i \cdot \delta P_i + \sum \theta_i \cdot \delta M_i$$

$$\delta_{Ey} \cdot 1 = \sum \left(\frac{PL}{EA} \right)_i \cdot \delta P_i + \sum \int_0^L \left[\frac{M_x}{EI} \right] \cdot \delta M_x \, dx$$

However, since $\delta P = 0$, we only need calculate the second term:





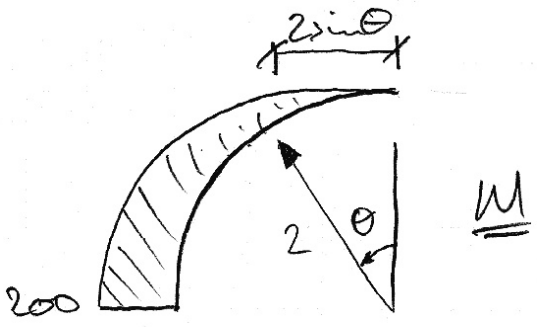
For *AB* we have:

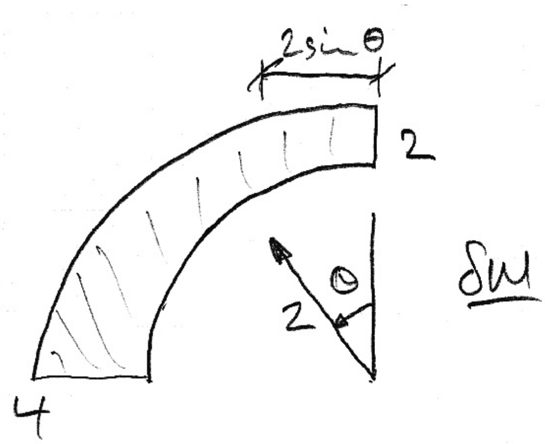
$$\int_A^B \left[\frac{M_x}{EI} \right] \cdot \delta M_x \, dx = \frac{1}{EI} \left[\frac{1}{2} (200 + 142.8)(4)(2) \right] = \frac{1371.2}{EI}$$

For *BC* we have:

$$\int_B^C \left[\frac{M_x}{EI} \right] \cdot \delta M_x \, dx = \frac{1}{EI} \left[(200)(4)(2) \right] = \frac{1600}{EI}$$

For *CD*, we have the following equations for the bending moments:





$$M(\theta) = (100)(2\sin\theta)$$
$$= 200\sin\theta$$

$$\delta M(\theta) = 2 + (1)(2\sin\theta)$$
$$= 2 + 2\sin\theta$$

Also note that we want to integrate around the member – an integrand ds - but only have the moment expressed according to θ, we must change the integration limits by substituting:

$$ds = R \cdot d\theta = 2d\theta$$

Thus we have:

$$\begin{aligned}\int_C^D \left[\frac{M_x}{EI}\right] \cdot \delta M_x \, dx &= \frac{1}{EI}\int_0^{\pi/2} (200\sin\theta)(2 + 2\sin\theta) \cdot 2d\theta \\ &= \frac{800}{EI}\int_0^{\pi/2} \left(\sin\theta + \sin^2\theta\right) d\theta \\ &= \frac{800}{EI}\left[\int_0^{\pi/2} \sin\theta \, d\theta + \int_0^{\pi/2} \sin^2\theta \, d\theta\right]\end{aligned}$$

Taking each term in turn:

$$\int_0^{\pi/2} \sin\theta \, d\theta = \left[-\cos\theta\right]_0^{\pi/2} = -0 - (-1) = +1$$

$$\int_0^{\pi/2} \sin^2\theta \, d\theta = \left[\frac{\theta}{2} - \frac{1}{4}\sin^2\theta\right]_0^{\pi/2} = \left[\frac{\pi}{4} - \frac{1}{4}(1)^2\right] - \left[0 - \frac{1}{4}(0)^2\right] = \frac{\pi - 1}{4}$$

Thus:

$$\int_C^D \left[\frac{M_x}{EI}\right] \cdot \delta M_x \, dx = \frac{800}{EI}\left(1 + \frac{\pi - 1}{4}\right) = \frac{200\pi + 600}{EI}$$

Thus:

$$\delta_{Ey} = \frac{1371.2}{EI} + \frac{1600}{EI} + \frac{200\pi + 600}{EI} = +\frac{4200}{EI}$$

Thus we get a downwards deflection as expected. Also, since $EI = 120 \times 10^3$ kNm^2, we have:

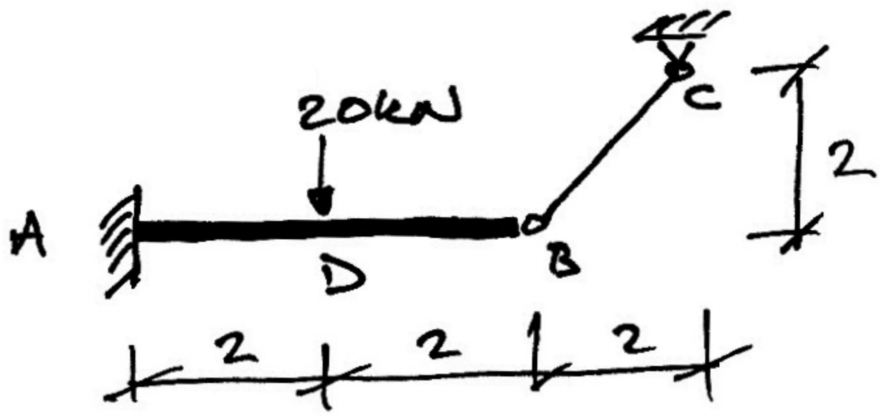
$$\delta_{Ey} = \frac{4200}{120 \times 10^3} = 35 \text{ mm} \downarrow$$

2.3.5 Problems

Problem 1

For the following structure, find the BMD and the vertical deflection at *D*. Take $EI = 8\times10^3 \text{kNm}^2$ and $EA = 16\times10^3$ kN.

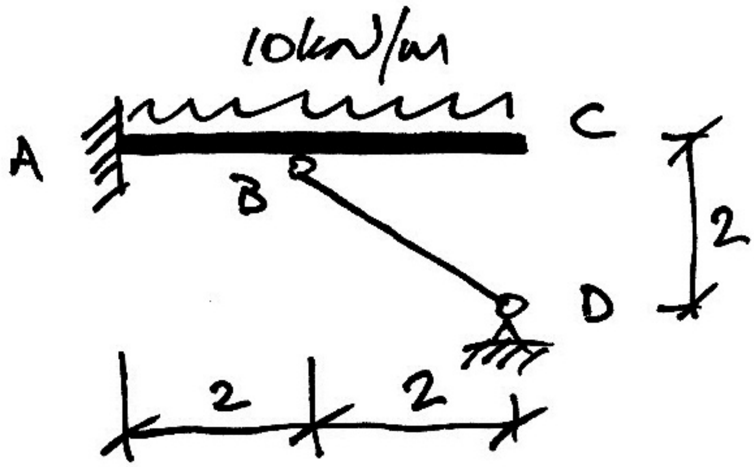
(*Ans.* $\alpha = 7.8$ for *BC*, $\delta_{By} = 1.93$ mm $\downarrow$)



Problem 2

For the following structure, find the BMD and the vertical deflection at *C*. Take $EI = 8\times10^3 \text{kNm}^2$ and $EA = 16\times10^3$ kN.

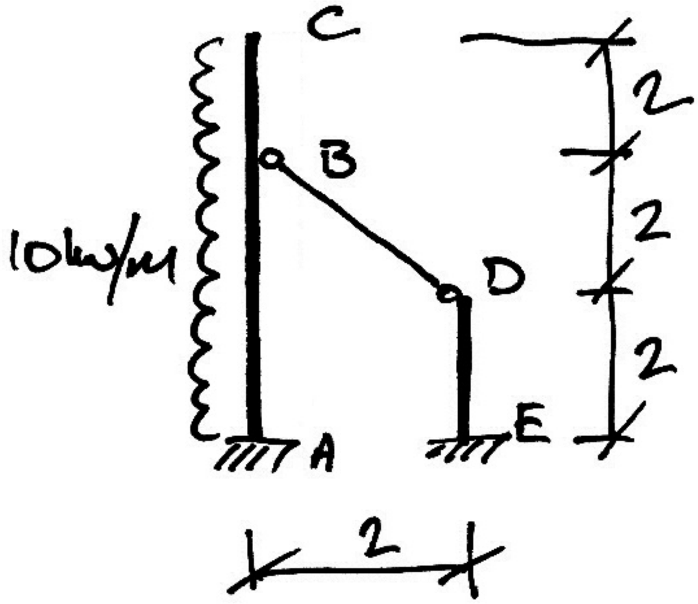
(*Ans.* $\alpha = 25.7$ for *BD*, $\delta_{Cv} = 25$ mm $\downarrow$)



Problem 3

For the following structure, find the BMD and the horizontal deflection at *C*. Take $EI = 8\times10^3\,\mathrm{kNm}^2$ and $EA = 16\times10^3\ \mathrm{kN}$.

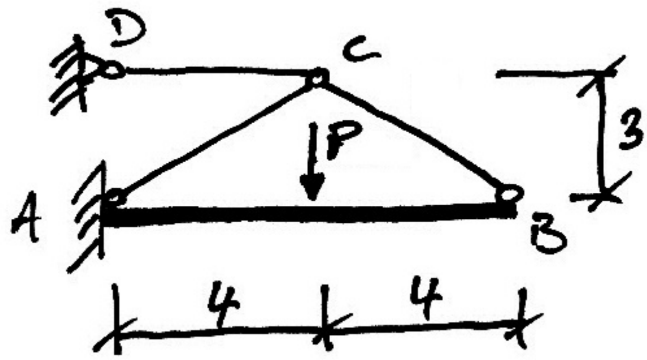
(*Ans.* $\alpha = 47.8$ for *BD*, $\delta_{Cx} = 44.8\ \mathrm{mm}\rightarrow$)



Problem 4

For the following structure, find the BMD and the vertical deflection at *B*. Take *P* = 20 kN, $EI = 8\times10^3\,\mathrm{kNm}^2$ and $EA = 16\times10^3\ \mathrm{kN}$.

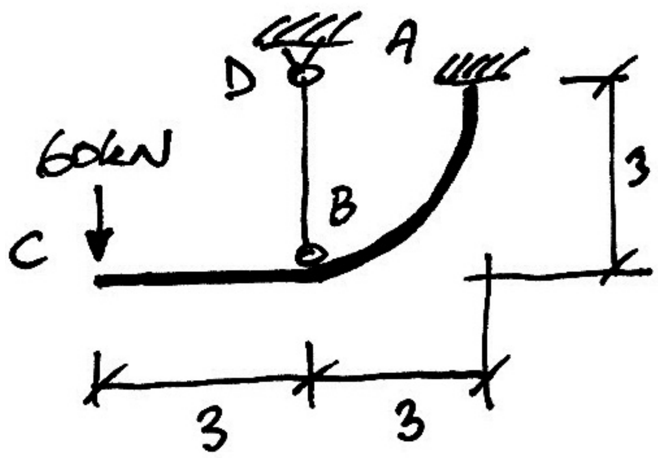
(*Ans*. $\alpha = 14.8$ for *CD*, $\delta_{By} = 14.7$ mm $\downarrow$)



Problem 5

For the following structure, find the BMD and the vertical deflection at *C*. Take $EI = 50 \times 10^3 \text{kNm}^2$ and $EA = 20 \times 10^3$ kN.

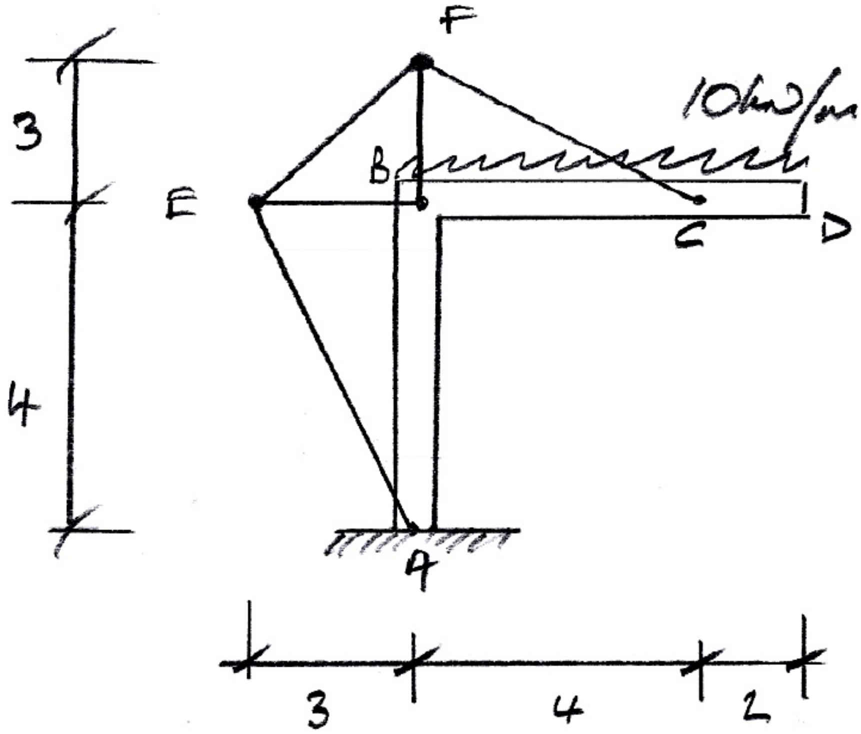
(*Ans*. $\alpha = 100.5$ for *BC*, $\delta_{Cy} = 55.6$ mm $\downarrow$)



Problem 6

Analyze the following structure and determine the BMD and the vertical deflection at *D*. For *ABCD*, take $E = 10$ kN/mm^2, $A = 12 \times 10^4$ mm^2 and $I = 36 \times 10^8$ mm^4, and for *AEBFC* take $E = 200$ kN/mm^2 and $A = 2 \times 10^3$ mm^2.

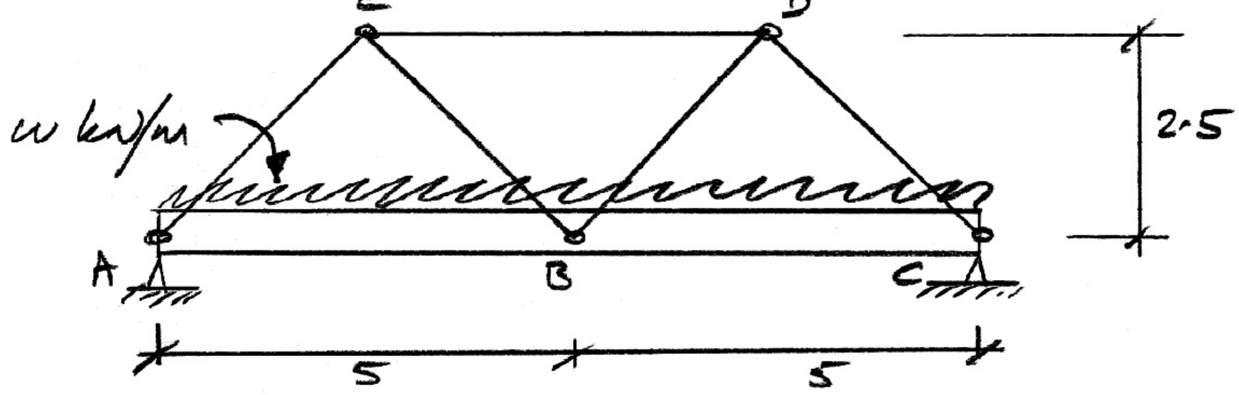
(*Ans*. $\alpha = 109.3$ for *BF*, $\delta_{Cy} = 54.4$ mm $\downarrow$)



Problem 7

Analyze the following structure. For all members, take $E = 10$ kN/mm^2, for *ABC*, $A = 6\times10^4$ mm^2 and $I = 125\times10^7$ mm^4; for all other members $A = 1000$ mm^2.

($Ans.$ $\alpha = 72.5$ for *DE*)



2.4 Past Exam Questions

2.4.1 Sample Paper 2007

3. For the rigidly jointed frame shown in Fig. Q3, using Virtual Work:

(i) Determine the bending moment moments due to the loads as shown;
(*15 marks*)

(ii) Draw the bending moment diagram, showing all important values;
(*4 marks*)

(iii) Determine the reactions at *A* and *E*;
(*3 marks*)

(iv) Draw the deflected shape of the frame.
(*3 marks*)

Neglect axial effects in the flexural members.
Take the following values:
I for the frame = 150×10^6 mm^4;
Area of the stay *EB* = 100 mm^2;
Take *E* = 200 kN/mm^2 for all members.

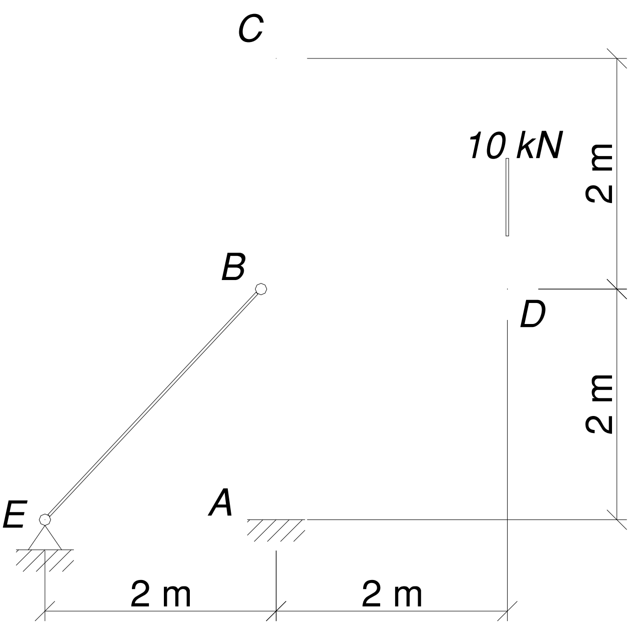


FIG. Q3

2.4.2 Semester 1 Exam 2007

3. For the rigidly jointed frame shown in Fig. Q3, using Virtual Work:

(i) Determine the bending moment moments due to the loads as shown;

(*15 marks*)

(ii) Draw the bending moment diagram, showing all important values;

(*4 marks*)

(iii) Determine the reactions at *A* and *E*;

(*3 marks*)

(iv) Draw the deflected shape of the frame.

(*3 marks*)

Neglect axial effects in the flexural members.
Take the following values:
I for the frame = 150×10^6 mm^4;
Area of the stay *EF* = 200 mm^2;
Take *E* = 200 kN/mm^2 for all members.

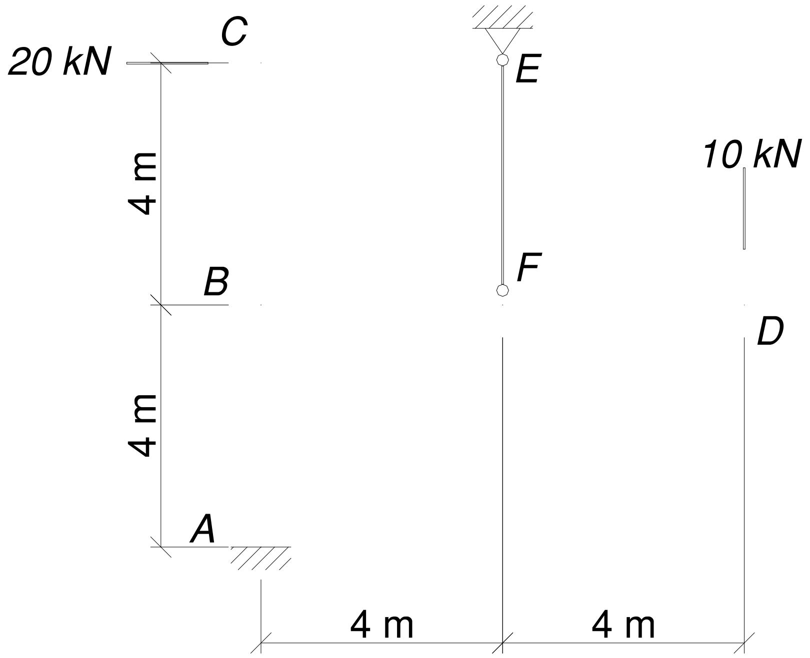
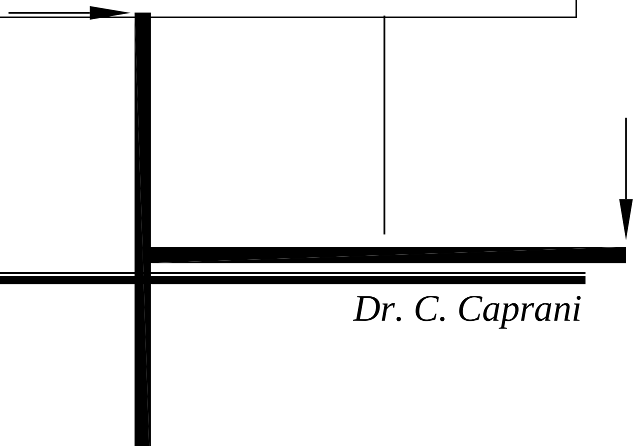


FIG. Q3

Ans. $\alpha = 35.0$.



2.4.3 Semester 1 Exam 2008

QUESTION 3

For the frame shown in Fig. Q3, using Virtual Work:

(i) Determine the force in the tie;
(ii) Draw the bending moment diagram, showing all important values;
(iii) Determine the deflection at *C*;
(iv) Determine an area of the tie such that the bending moments in the beam are minimized;
(v) For this new area of tie, determine the deflection at *C*;
(vi) Draw the deflected shape of the structure.

(25 marks)

Note:

Neglect axial effects in the flexural members and take the following values:

- For the frame, $I = 600 \times 10^6\ \text{mm}^4$;
- For the tie, $A = 300\ \text{mm}^2$;
- For all members, $E = 200\ \text{kN/mm}^2$.

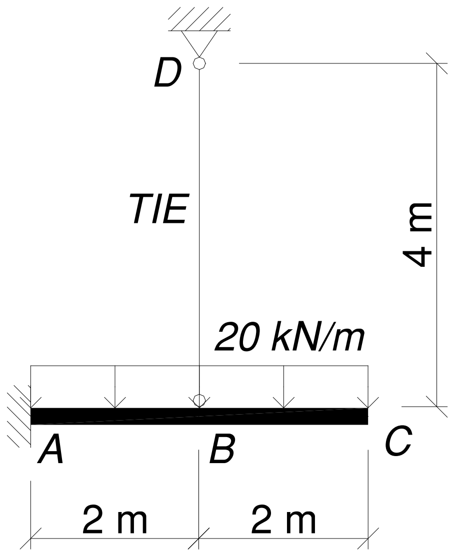


FIG. Q3

Ans. $\alpha = 21.24$; $\delta_{Cy} = 4.1\ \text{mm} \downarrow$; $A = 2160\ \text{mm}^2$; $\delta_{Cy} = 2.0\ \text{mm} \downarrow$

2.4.4 Semester 1 Exam 2009

QUESTION 3

For the frame shown in Fig. Q3, using Virtual Work:

(i) Determine the axial forces in the members;
(ii) Draw the bending moment diagram, showing all important values;
(iii) Determine the reactions;
(iv) Determine the vertical deflection at *D*;
(v) Draw the deflected shape of the structure.

(25 marks)

Note:

Neglect axial effects in the flexural members and take the following values:

- For the beam *ABCD*, $I = 600\times10^6\ \text{mm}^4$;
- For members *BF* and *CE*, $A = 300\ \text{mm}^2$;
- For all members, $E = 200\ \text{kN/mm}^2$.

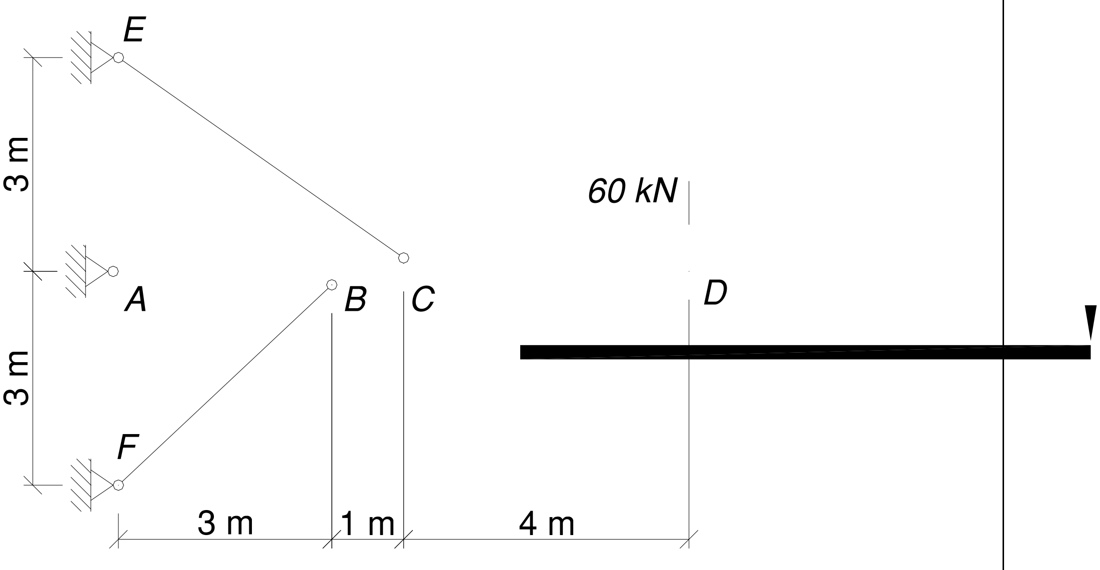


FIG. Q3

Ans. $\alpha = 113.7$ (for CE); $\delta_{Dy} = 55\ \text{mm}\downarrow$

2.4.5 Semester 1 Exam 2010

QUESTION 3

For the frame shown in Fig. Q3, using Virtual Work:

(i) Draw the bending moment diagram, showing all important values;

(ii) Determine the horizontal displacement at *C*;

(iii) Determine the vertical deflection at *C*;

(iv) Draw the deflected shape of the structure.

(25 marks)

Note:
Neglect axial effects in the flexural members and take the following values:

- For the beam *ABC*, $EI = 5\times10^3 \text{ kNm}^2$;
- For member *BD*, $E = 200 \text{ kN/mm}^2$ and $A = 200 \text{ mm}^2$;
- The following integral results may assist in your solution:

$$\int \sin\theta \, d\theta = -\cos\theta \qquad \int \cos\theta \sin\theta \, d\theta = -\frac{1}{4}\cos 2\theta \qquad \int \sin^2\theta \, d\theta = \frac{\theta}{2} - \frac{1}{4}\sin 2\theta$$

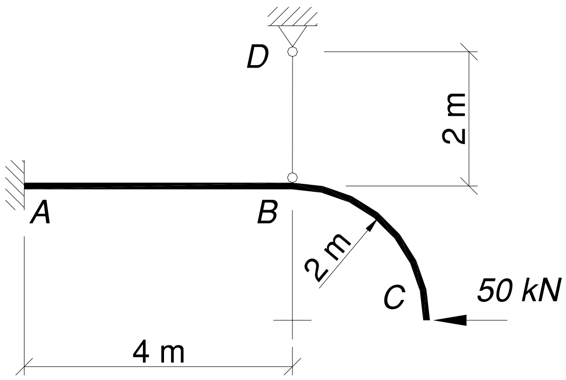


FIG. Q3

Ans. $\alpha = 37.1$ (for *BD*); $\delta_{Cx} = 104 \text{ mm} \leftarrow$ $\delta_{Cy} = 83 \text{ mm} \downarrow$

2.4.6 Semester 1 Exam 2011

QUESTION 3

For the frame shown in Fig. Q3, using Virtual Work:
(i) Draw the bending moment diagram, showing all important values;
(ii) Draw the axial force diagram;
(iii) Determine the vertical deflection at *D*;
(iv) Draw the deflected shape of the structure.

(25 marks)

Note:
Neglect axial effects in the flexural members and take the following values:

- For the member *ABCD*, $EI = 5\times10^3\ \text{kNm}^2$;
- For members *BF* and *CE*, $E = 200\ \text{kN/mm}^2$ and $A = 200\ \text{mm}^2$;
- The following integral result may assist in your solution:

$$\int \sin^2\theta\, d\theta = \frac{\theta}{2} - \frac{1}{4}\sin 2\theta$$

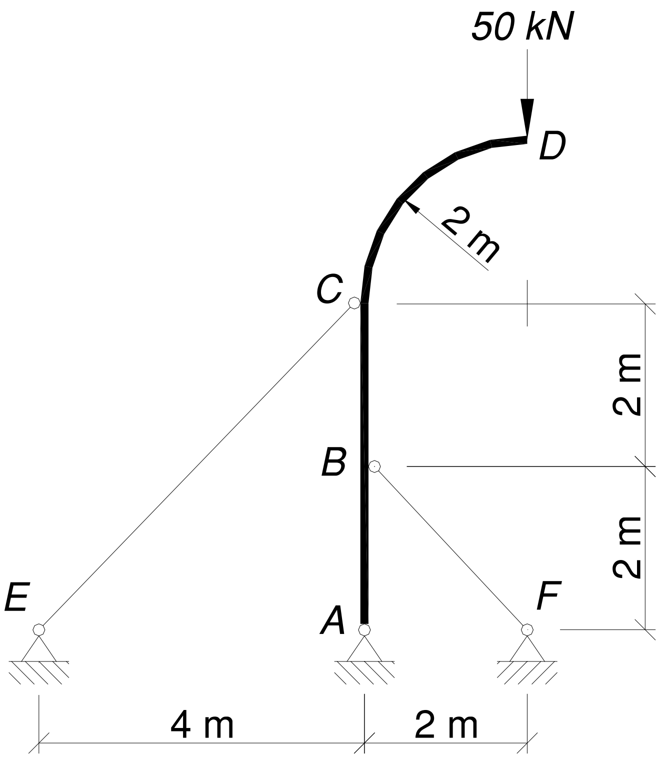


FIG. Q3

Ans. $\alpha = 48.63$ (for *BF*); $\delta_{Dy} = 108.4\ \text{mm}\downarrow$

2.5 Appendix – Trigonometric Integrals

2.5.1 Useful Identities

In the following derivations, use is made of the trigonometric identities:

$$\cos\theta\sin\theta = \frac{1}{2}\sin 2\theta \tag{1}$$

$$\cos^2\theta = \frac{1}{2}\left(1+\cos 2\theta\right) \tag{2}$$

$$\sin^2\theta = \frac{1}{2}\left(1-\cos 2\theta\right) \tag{3}$$

Integration by parts is also used:

$$\int u\,dx = ux - \int x\,du + C \tag{4}$$

2.5.2 Basic Results

Neglecting the constant of integration, some useful results are:

$$\int \cos\theta \, d\theta = \sin\theta \tag{5}$$

$$\int \sin\theta \, d\theta = -\cos\theta \tag{6}$$

$$\int \sin a\theta \, d\theta = -\frac{1}{a}\cos a\theta \tag{7}$$

$$\int \cos a\theta \, d\theta = \frac{1}{a}\sin a\theta \tag{8}$$

2.5.3 Common Integrals

The more involved integrals commonly appearing in structural analysis problems are:

$\int \cos\theta \sin\theta \, d\theta$

Using identity (1) gives:

$$\int \cos\theta \sin\theta \, d\theta = \frac{1}{2}\int \sin 2\theta \, d\theta$$

Next using (7), we have:

$$\begin{aligned}\frac{1}{2}\int \sin 2\theta \, d\theta &= \frac{1}{2}\left[-\frac{1}{2}\cos 2\theta\right] \\ &= -\frac{1}{4}\cos 2\theta\end{aligned}$$

And so:

$$\int \cos\theta \sin\theta \, d\theta = -\frac{1}{4}\cos 2\theta \qquad (9)$$

$\int \cos^2 \theta \, d\theta$

Using (2), we have:

$$\begin{aligned}\int \cos^2 \theta \, d\theta &= \frac{1}{2}\int \left(1+\cos 2\theta\right) d\theta \\ &= \frac{1}{2}\left[\int 1 d\theta + \int \cos 2\theta d\theta\right]\end{aligned}$$

Next using (8):

$$\begin{aligned}\frac{1}{2}\left[\int 1 d\theta + \int \cos 2\theta d\theta\right] &= \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right] \\ &= \frac{\theta}{2} + \frac{1}{4}\sin 2\theta\end{aligned}$$

And so:

$$\int \cos^2 \theta \, d\theta = \frac{\theta}{2} + \frac{1}{4}\sin 2\theta \qquad (10)$$

$\int \sin^2 \theta \, d\theta$

Using (3), we have:

$$\begin{aligned}\int \sin^2 \theta \, d\theta &= \frac{1}{2}\int \left(1-\cos 2\theta\right) d\theta \\ &= \frac{1}{2}\left[\int 1 d\theta - \int \cos 2\theta d\theta\right]\end{aligned}$$

Next using (8):

$$\begin{aligned}\frac{1}{2}\left[\int 1 d\theta - \int \cos 2\theta d\theta\right] &= \frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right] \\ &= \frac{\theta}{2} - \frac{1}{4}\sin 2\theta\end{aligned}$$

And so:

$$\int \sin^2 \theta \, d\theta = \frac{\theta}{2} - \frac{1}{4}\sin 2\theta \tag{11}$$

$\int \theta \cos\theta \, d\theta$

Using integration by parts write:

$$\int \theta \cos\theta \, d\theta = \int u \, dx$$

Where:

$$u = \theta \qquad\qquad dx = \cos\theta \, d\theta$$

To give:

$$du = d\theta$$

And

$$\begin{aligned} \int dx &= \int \cos\theta \, d\theta \\ x &= \sin\theta \end{aligned}$$

Which uses (5). Thus, from (4), we have:

$$\begin{aligned} \int u \, dx &= ux - \int x \, du \\ \int \theta \cos\theta \, d\theta &= \theta \sin\theta - \int \sin\theta \, d\theta \end{aligned}$$

And so, using (6) we have:

$$\int \theta \cos\theta \, d\theta = \theta \sin\theta + \cos\theta \tag{12}$$

$\int \theta \sin\theta \, d\theta$

Using integration by parts write:

$$\int \theta \sin\theta \, d\theta = \int u \; dx$$

Where:

$$u = \theta \qquad\qquad dx = \sin\theta \; d\theta$$

To give:

$$du = d\theta$$

And

$$\int dx = \int \sin\theta \; d\theta$$
$$x = -\cos\theta$$

Which uses (6). Thus, from (4), we have:

$$\int u \; dx = ux - \int x \; du$$
$$\int \theta \sin\theta \, d\theta = \theta\left(-\cos\theta\right) - \int \left(-\cos\theta\right) \, d\theta$$

And so, using (5) we have:

$$\int \theta \sin\theta \, d\theta = -\theta \cos\theta + \sin\theta \tag{13}$$

$\int \cos(A-\theta)\,d\theta$

Using integration by substitution, we write $u = A - \theta$ to give:

$$\frac{du}{d\theta} = -1$$
$$du = -d\theta$$

Thus:

$$\int \cos(A-\theta)\,d\theta = \int \cos u\,(-du)$$

And since, using (5):

$$-\int \cos u\; du = -\sin u$$

We have:

$$\int \cos(A-\theta)\,d\theta = -\sin(A-\theta) \tag{14}$$

$\int \sin(A-\theta)\,d\theta$

Using integration by substitution, we write $u = A - \theta$ to give:

$$\frac{du}{d\theta} = -1$$
$$du = -d\theta$$

Thus:

$$\int \sin(A-\theta)\,d\theta = \int \sin u\,(-du)$$

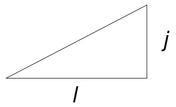
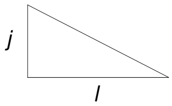
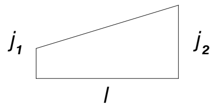
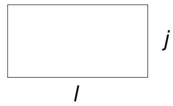
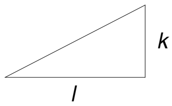
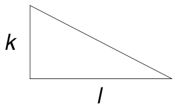
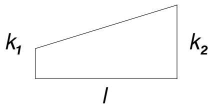
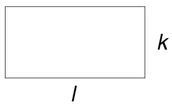
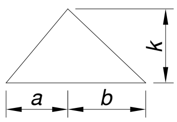
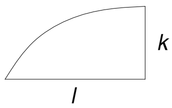
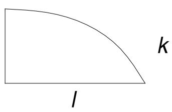
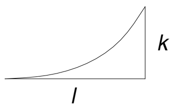
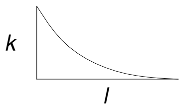
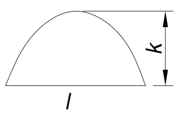
And since, using (6):

$$-\int \sin u\; du = -(-\cos u)$$

We have:

$$\int \sin(A-\theta)\,d\theta = \cos(A-\theta) \tag{15}$$

2.6 Appendix – Volume Integrals

				
	$\frac{1}{3}jkl$	$\frac{1}{6}jkl$	$\frac{1}{6}(j_1+2j_2)kl$	$\frac{1}{2}jkl$
	$\frac{1}{6}jkl$	$\frac{1}{3}jkl$	$\frac{1}{6}(2j_1+j_2)kl$	$\frac{1}{2}jkl$
	$\frac{1}{6}j(k_1+2k_2)l$	$\frac{1}{6}j(2k_1+k_2)l$	$\frac{1}{6}\left[j_1(2k_1+k_2)+ j_2(k_1+2k_2)\right]l$	$\frac{1}{2}j(k_1+k_2)l$
	$\frac{1}{2}jkl$	$\frac{1}{2}jkl$	$\frac{1}{2}(j_1+j_2)kl$	jkl
	$\frac{1}{6}jk(l+a)$	$\frac{1}{6}jk(l+b)$	$\frac{1}{6}\left[j_1(l+b)+ j_2(l+a)\right]k$	$\frac{1}{2}jkl$
	$\frac{5}{12}jkl$	$\frac{1}{4}jkl$	$\frac{1}{12}(3j_1+5j_2)kl$	$\frac{2}{3}jkl$
	$\frac{1}{4}jkl$	$\frac{5}{12}jkl$	$\frac{1}{12}(5j_1+3j_2)kl$	$\frac{2}{3}jkl$
	$\frac{1}{4}jkl$	$\frac{1}{12}jkl$	$\frac{1}{12}(j_1+3j_2)kl$	$\frac{1}{3}jkl$
	$\frac{1}{12}jkl$	$\frac{1}{4}jkl$	$\frac{1}{12}(3j_1+j_2)kl$	$\frac{1}{3}jkl$
	$\frac{1}{3}jkl$	$\frac{1}{3}jkl$	$\frac{1}{3}(j_1+j_2)kl$	$\frac{2}{3}jkl$